Exam MPRI

2021

Note:

• Lecture notes are allowed: using theorems proved during the lectures is expected.
• Although they look similar, the exercises 1 and 2 are independent, and answers from one are not useful or required for the other one.

Definitions for Exercises 1 and 2

This is just a reminder, this is the definitions used in Nathanaël Fijalkow’s lectures.

We consider two-player deterministic finite games. An arena \( A \) is given by a set \( V \) of vertices with \( V = V_{\text{Eve}} \cup V_{\text{Adam}} \) and a set \( E \subseteq V \times V \) of edges. We make the assumption that every vertex has at least one outgoing edge. A winning condition for \( A \) is \( W \subseteq V^\omega \). A game \( G \) is a pair \( (A, W) \).

A strategy for Eve is \( \sigma : V^* \cdot V_{\text{Eve}} \to E \), and for Adam \( \tau : V^+ \cdot V_{\text{Adam}} \to E \). A path is a sequence \( v_0v_1 \ldots \) such that for all \( i \) we have \((v_i, v_{i+1}) \in E \). It is consistent with \( \sigma \) if for all \( i \), if \( v_i \in V_{\text{Eve}} \) then \( \sigma(v_0 \ldots v_i) = (v_i, v_{i+1}) \). The strategy \( \sigma \) is winning from \( v \in V \) if all infinite paths \( \pi \) from \( v \) consistent with \( \sigma \) satisfy \( W \), meaning \( \pi \in W \). In that case we say that \( v \) is winning for Eve. Symmetrically we define \( v \) being winning for Adam.

We say that \( G \) is determined if for all \( v \in V \), either \( v \) is winning for Eve or \( v \) is winning for Adam. All games we consider are determined (Martin’s theorem says that it holds for any Borel objective): we use this result without proving it. We write \( W_{\text{Eve}}(G) \) for the set of winning vertices for Eve, and \( W_{\text{Adam}}(G) \) for Adam. Then \( G \) is determined if \( W_{\text{Eve}}(G) \cup W_{\text{Adam}}(G) = V \).

A positional strategy for Eve is \( \sigma : V_{\text{Eve}} \to E \), and for Adam \( \tau : V_{\text{Adam}} \to E \). We say that \( G \) is positionally determined for Eve if for all \( v \in W_{\text{Eve}}(G) \), there exists a positional winning strategy from \( v \). Similarly for Adam.

An objective is \( \Omega \subseteq C^\omega \) with \( C \) a set of colours. The objective \( \Omega \) and a colouring function \( \text{col} : V \to C \) (we colour vertices) induce a condition \( \Omega[\text{col}] \subseteq V^\omega \):

\[
\Omega[\text{col}] = \{v_0v_1 \cdots : \text{col}(v_0)\text{col}(v_1) \cdots \in \Omega\}.
\]

We say that \( G = (A, \Omega[\text{col}]) \) has objective \( \Omega \), and that:

• \( \Omega \) is prefix independent if for all \( w \in C^*, w' \in C^\omega \) we have \( w' \in \Omega \iff ww' \in \Omega \).
• \( \Omega \) is positionally determined for Eve if all games with objectives \( \Omega \) are positionally determined.
• \( \Omega \) is positionally determined if it holds for both Eve and Adam.

In evaluating algorithms the important parameters from the graph are \( n \) the number of vertices and \( m \) the number of edges.
Exercise 1

Let $C = [1, d]$ for $d \in \mathbb{N}$. We define the weak parity objective:

$$\text{WeakParity} = \{ \rho \in [1, d]^\omega : \max(\rho) \text{ is even} \}.$$

**Question 1:** Prove or disprove: WeakParity is prefix independent.

Let $G$ be a game with objective WeakParity: $G = (A, \text{WeakParity}[\text{col}])$. Let $V_d = \{ v \in V : \text{col}(v) = d \}$. Let us assume that $d$ is even.

**Question 2:** Show that if Attr$_{Eve}(V_d) = V$, then $W_{Eve}(G) = V$.

For $F \subseteq V$ we define the reachability condition Reach$(F) = \{ v_0 v_1 \cdots : \exists i \in \mathbb{N}, v_i \in F \}$. We write Attr$_{Eve}(F)$ for $W_{Eve}(A, \text{Reach}(F))$.

**Question 3:** Define the game induced from $G$ by $V \setminus \text{Attr}_{Eve}(d)$ and show that in this induced game, every vertex has at least one outgoing edge (so it is well defined).

**Question 4:** Let $G'$ the game induced from $G$ by $V \setminus \text{Attr}_{Eve}(d)$. Show that $W_{Eve}(G) = \text{Attr}_{Eve}(V_d) \cup W_{Eve}(G')$.

**Question 5:** Construct an algorithm for solving weak parity games (meaning computing the set of winning vertices for Eve) and evaluate its complexity.

**Question 6:** Prove or disprove (for both players): WeakParity is positionally determined.

Exercise 2

Let $C = [1, k]$ for $k \in \mathbb{N}$. We define the generalized CoBüchi objective:

$$\text{GenCoBuchi} = \{ \rho \in [1, k]^\omega : \exists i \in [1, k], i \notin \text{inf}(\rho) \} ,$$

where $\text{inf}(\rho)$ is the set of colours appearing infinitely many times in $\rho$.

**Question 1:** Prove or disprove: GenCoBuchi is prefix independent.

**Question 2:** Prove or disprove (for both players): GenCoBuchi is positionally determined.

Let $G$ be a game with objective GenCoBuchi: $G = (A, \text{GenCoBuchi}[\text{col}])$. We write $V_i = \{ v \in V : \text{col}(v) = i \}$, and for $F \subseteq V$ we write CoBuchi$(F) = \{ \pi \in V^\omega : \text{inf}(\rho) \cap F = \emptyset \}$.

**Question 3:** Prove or disprove: $W_{Eve}(G) = \bigcup_{i \in [1, k]} W_{Eve}(A, \text{CoBuchi}(V_i))$.

**Question 4:** Show that if for all $i \in [1, k]$ we have $W_{Eve}(A, \text{CoBuchi}(V_i)) = \emptyset$, then $W_{Eve}(G) = \emptyset$.

**Question 5:** Assume that for some $i \in [1, k]$ we have $W_{Eve}(A, \text{CoBuchi}(V_i)) \neq \emptyset$, show that the game $G'$ induced from $G$ by $V \setminus W_{Eve}(A, \text{CoBuchi}(V_i))$ is well defined, and that $W_{Eve}(G) = W_{Eve}(G') \cup W_{Eve}(A, \text{CoBuchi}(V_i))$.

**Question 6:** Construct an algorithm for solving generalized CoBüchi games (meaning computing the set of winning vertices for Eve) and evaluate its complexity.