**UNIVERSAL TREES**

(n, h)-tree:
- rooted
- arbitrary degree
- children are ordered
- all leaves have depth h
- at most m leaves

Size: number of leaves

(9, 2)-tree

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The red tree embeds into the white tree

A tree is (n, h)-universal if it embeds all (n, h)-trees

A (5, 2)-universal tree
Upper bounds

An exponential construction:

\[ n \]

it has \( n^h \) leaves

\[ n = 3 \quad h = 2 \]

The \((n,h)\)-complete tree

A quasi-polynomial construction:

\[ C(n,h) \]

\[ C(n/2, h) \quad C(n, h-1) \quad C(n/2, h) \]
We show that $C(n,h)$ is $(m,h)$-universal.

Let $t$ an $(m,h)$-tree:

$|t_1| + |t_2| + \ldots + |t_k| \leq m$

$\Rightarrow \exists i, \quad |t_1| + \ldots + |t_{i-1}| \leq m/2$

$|t_1| + \ldots + |t_i| > m/2$

implying $|t_{i+1}| + \ldots + |t_k| \leq m/2$
Size of $\mathcal{C}(n, h)$:

$$|\mathcal{C}(n, h)| = |\mathcal{C}(\frac{n}{2}, h)| + |\mathcal{C}(n, h-1)| + |\mathcal{C}(\frac{n}{2}, h)|$$

$$\mathcal{C}(n, h) \leq n \cdot \left( h + \log(n) \right)$$

$$\begin{cases} \sim n \log h \quad \text{quasipolynomial} \\ \sim n^5 \quad \text{for } h = O(\log n) \end{cases}$$
Lower bounds

All $(n, h)$-universal trees have size at least $g(n, h)$ where:

$$g(n, h) = \sum_{\delta=1}^{n} g(\lfloor n/\delta \rfloor, h-1)$$

$$g(n, h) = n \Omega(\log(h))$$

$$\frac{|\mathcal{C}(n, h)|}{g(n, h)} = O(nh)$$

**Conjecture:** C is optimal.