Homework assignment MPRI 2023

Deadline: 24 November 2023 AOE

1 Dominions

Recall that we assume that in our games every vertex has an outgoing edge. This is for technical convenience, as it avoids defining what happens to finite plays. Given a game \( \mathcal{G} \), the subgame induced by a set of vertices \( X \) is defined in the natural way and written \( \mathcal{G}[X] \). However, it is only well defined if for every vertex in \( X \), there exists an outgoing edge to \( X \), to satisfy the assumption of games. For \( X \) a subset of vertices, let us write \( \text{Attr}_{\text{Eve}}(X, \mathcal{G}) \) for the attractor of Eve to \( X \) in \( \mathcal{G} \).

For any \( X \), the set \( \mathcal{G}[\text{Attr}_{\text{Eve}}(X, \mathcal{G})] \) induces a subgame, which we write \( \mathcal{G}[\text{Attr}_{\text{Eve}}(X, \mathcal{G})] \).

Let us introduce two more definitions:

• We say that \( X \) is closed for Eve in \( \mathcal{G} \) if \( \mathcal{G} \) is closed for Eve in \( \mathcal{G} \). Equivalently, every vertex of Eve in \( X \) has at least one outgoing edge to \( X \), and every vertex of Adam in \( X \) has all outgoing edges to \( X \). In words, Eve can ensure to stay in \( X \).

• A dominion for Eve in \( \mathcal{G} \) is a subset \( D \) of vertices such that for every vertex of \( D \), Eve has a strategy which is both winning and ensuring that all plays remain in \( D \) forever.

All definitions are given using Eve’s point of view but easily adapted to Adam’s.

**Question 1** (2 drawings). Prove by drawing the following two properties. Let \( D \) be a dominion for Eve in \( \mathcal{G} \), and \( X \) a subset of vertices.

• If \( D \) does not intersect with \( X \), then \( D \) is also a dominion for Eve in \( \mathcal{G}[\text{Attr}_{\text{Adam}}(X, \mathcal{G})] \).

• If \( X \) is closed for Adam in \( \mathcal{G} \), then \( D \cap X \) is a dominion for Eve in \( \mathcal{G}[X] \).

2 The algorithm

We spell out the pseudocode of \( \text{SolveEven} \) in Algorithm 1, leaving out the symmetric \( \text{SolveOdd} \).

**Algorithm 1**: The algorithm for computing the winning region of parity games.

```plaintext
Data: A parity game \( \mathcal{G} \) with priorities in \([1, d]\)
Function \( \text{SolveEven}(\mathcal{G}, d, s_{\text{Eve}}, s_{\text{Adam}}) \):
  if \( \mathcal{G} \) is empty or \( s_{\text{Adam}} = 0 \) then
    return \( V(\mathcal{G}) \)
  \( G_1 \leftarrow \text{SolveOdd}(\mathcal{G}, d, s_{\text{Eve}}, \lfloor s_{\text{Adam}}/2 \rfloor) \)
  \( V_d \leftarrow \{ v \in V(G_1) : \text{col}(v) = d \} \)
  \( H \leftarrow G_1 \setminus \text{Attr}_{\text{Eve}}(V_d, G_1) \)
  \( W_{\text{Adam}} \leftarrow \text{SolveOdd}(H, d - 1, s_{\text{Eve}}, s_{\text{Adam}}) \)
  \( G_2 \leftarrow G_1 \setminus \text{Attr}_{\text{Adam}}(W_{\text{Adam}}, G_1) \)
  \( G_3 \leftarrow \text{SolveEven}(G_2, d, s_{\text{Eve}}, \lfloor s_{\text{Adam}}/2 \rfloor) \)
  return \( V(G_3) \)
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Question 2 (1000 words). A picture is worth a thousand words: make a good drawing explaining the algorithm.

We will prove the following properties of the algorithm.

- If $d$ is even, then $\text{SolveEven}(G, d, s_{\text{Eve}}, s_{\text{Adam}})$ is closed for Eve in $G$ and:
  
  (i) contains all dominions for Eve in $G$ of size up to $s_{\text{Eve}},$
  
  (ii) does not intersect any dominion for Adam in $G$ of size up to $s_{\text{Adam}}.$

- If $d$ is odd, then $\text{SolveOdd}(G, d, s_{\text{Eve}}, s_{\text{Adam}})$ is closed for Adam in $G$ and:
  
  (i) contains all dominions for Adam in $G$ of size up to $s_{\text{Adam}},$
  
  (ii) does not intersect any dominion for Eve in $G$ of size up to $s_{\text{Eve}}.$

Question 3 (2 lines). Explain why this implies that $\text{SolveEven}(G, d, n, n) = W_{\text{Eve}}(G).$

Question 4 (1 line). Explain why it is enough to prove only the part about $\text{SolveEven}.$

The proof is by induction on $d + s_{\text{Eve}} + s_{\text{Adam}}.$

Question 5 (2 lines). Deal with the base cases. (Bonus: why is the $\text{SolveOdd}$ case not entirely symmetric?)

Question 6 (2 lines). Show that the algorithm terminates.

Question 7 (~10 lines). Show that the set returned by $\text{SolveEven}(G, d, s_{\text{Eve}}, s_{\text{Adam}})$ is closed for Eve in $G.$

Question 8 (~15 lines). Show the first property: the set returned by $\text{SolveEven}(G, d, s_{\text{Eve}}, s_{\text{Adam}})$ contains all dominions of Eve in $G$ of size up to $s_{\text{Eve}}.$

We proceed with showing the second property: the set returned by $\text{SolveEven}(G, d, s_{\text{Eve}}, s_{\text{Adam}})$ does not intersect with any dominion for Adam of size up to $s_{\text{Adam}}.$ Let $D$ be such a dominion.

We let $S$ be the union of all dominions for Adam in $G[D]$ of size up to $\lfloor s_{\text{Adam}}/2 \rfloor,$ and $A = \text{Attr}_{\text{Adam}}(S, G[D]).$ We distinguish two cases: $A = D$ and $A \neq D.$

Question 9 (~10 lines). Show that if $A = D,$ then $D$ does not intersect with $G_3$ and conclude.

We will now consider the case $A \neq D.$ Let us start with a lemma:

Lemma 1. Let $D$ be a non-empty dominion for Eve in $G.$ If all priorities in $D$ are at most $d$ and $d$ is odd, then there exists $D' \subseteq D$ a non-empty dominion for Eve in $G$ without vertices of priority $d.$

Question 10 (~5 lines). Prove the lemma.

Question 11 (~20 lines). Assume $A \neq D,$ show that $D \backslash A$ is a dominion in $G[D \backslash A],$ and use the lemma to conclude.

3 Complexity analysis

Question 12 (~10 lines). Write the recursive equations satisfied by the complexity of the algorithm as a function of $n$ number of vertices, $m$ number of edges, $d$ number of priorities, and $s_{\text{Eve}}, s_{\text{Adam}}.$ Compare with universal trees.