INTRODUCTION TO REINFORCEMENT LEARNING

Nathanaël FIJALKOW

$RL \subseteq ML$

ML:

Supervised learning:

Training data is labelled

Objective: predict unlabelled data

Unsupervised learning:

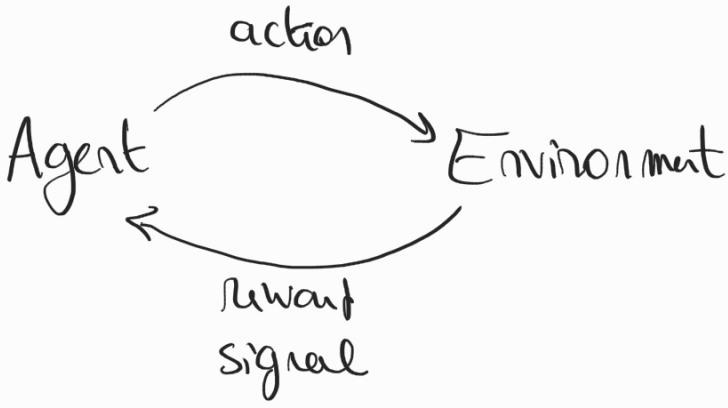
Training data is not labelled

objective: finding structure / pattern

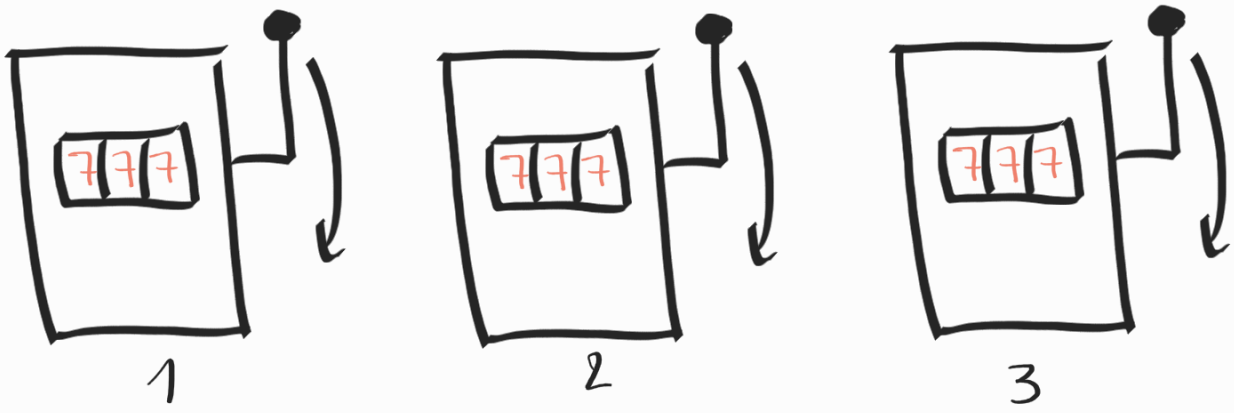
RL is neither supervised nor unsupervised !

RL: | active learning

| reward-based



MULTI ARMED BANDITS



arm = machine = action

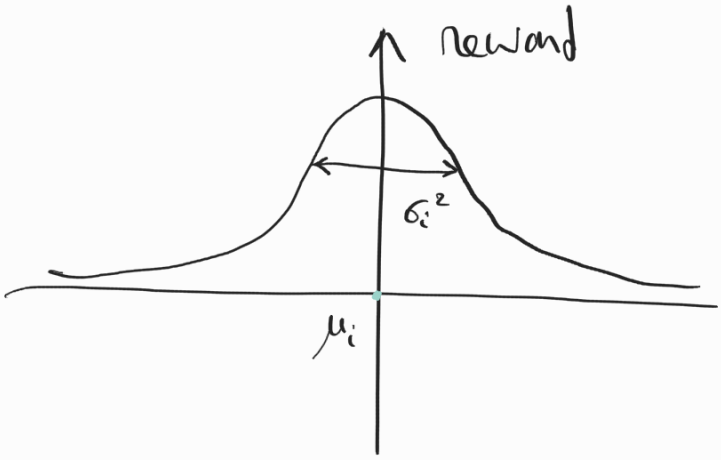
K machines : $1, \dots, K$

We assume that each machine i has a reward distribution δ_i

Bernoulli distribution $B(p)$

$$\begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Normal distribution $\mathcal{N}(\mu, \sigma^2)$



Notation: $\mu = \mathbb{E}[\delta]$ $\sigma^2 = \mathrm{Var}(\delta)$

Scenario:

- We know the number of machines but NOT THE REWARD DISTRIBUTIONS

- at each time step t we pick an arm i and draw a reward $r(t) \in \mathbb{R}$ from δ_i

- Two (similar) goals:

  → (1) identify the best arm: $\underset{i}{\text{argmax}}\ \mu_i$

  → (2) maximise the total reward:

$$\sum_{t \geq 1} r(t)$$ ← reward at time t

$K = 2$

$$\delta_1 = \begin{cases} 2 & \text{prob. } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}$$

$$\delta_2 = \begin{cases} 100 & \text{prob. } 1/10 \\ 0 & \text{prob } 9/10 \end{cases}$$

time	machine	
1	1 $\longrightarrow$	$r(1) = 2$
2	2 $\longrightarrow$	$r(2) = 0$
3	2 $\longrightarrow$	$r(3) = 0$
4	2 $\longrightarrow$	$r(4) = 0$

Difficulty: we have to make choices based on incomplete statistics!

Trade off between:

- getting **good information** on all machines:
  EXPLORATION
- getting good rewards
  EXPLOITATION

aftn T steps

for each machine $i \in [1, K]$

we have a set of samples

↳ $\hat{\mu}_i(T)$ empirical expectation

Greedy strategy:

play $\underset{i \in [1,K]}{\operatorname{argmax}} \hat{\mu}_i(T)$

EXPLOITATION

ε-Greedy strategy:

ε fixed $\varepsilon = 0.1$

exploration: uniformly at random over all actions — w. probability ε

exploitation: greedy: $\underset{i \in [1,K]}{\operatorname{argmax}} \hat{\mu}_i(T)$ — w. probability $1 - \varepsilon$

Notations

$$\arg\max_{i \in [1,k]} \mu_i \overset{\text{def}}{=} * \in [1,K] \text{ such that } \mu_* = \max_{i \in [1,K]} \mu_i$$

Regret analysis

Regret(T) = R(T) = difference between

"best a posteriori"

and

"what we achieved"

$$R(T) = T \cdot \mu_k - \sum_{t=1}^{T} r(t)$$

⚠ μ_i is the actual expectation

$\hat{\mu}_i(T)$ is the empirical expectation at time T

$$\max \sum_{t \geq 1}^{T} r(t)$$

$$\Longleftrightarrow$$

$$\min R(T)$$

We use the regret for comparing different strategies

So far:

Greedy

$$\operatorname*{argmax}_i \hat{\mu}_i(T)$$

ε-Greedy

$$\begin{cases} \operatorname*{argmax}_i \hat{\mu}_i(T) & \text{with probability } 1-\varepsilon \\ \text{uniform at random} & \text{with probability } \varepsilon \end{cases}$$

Issues:

- exploration never stops : at least ε is lost
- exploration does not take existing info into account
- may take a long time to converge

UPPER CONFIDENCE BOUNDS (UCB)

$r(t)$: reward at time t

$r(i,t)$: $\begin{cases} r(t) & \text{if } i \text{ was chosen at time } t \\ 0 & \text{o/w} \end{cases}$

$$R(T) = T \cdot \mu_* - \sum_{t=1}^{T} r(t)$$

$$\hat{\mu}_i(T) = \frac{1}{n(i,T)} \sum_{t=1}^{T} r(i,t)$$

UCB

$$\underset{i}{\operatorname{argmax}} \; \hat{\mu}_i(T) + c(i,T)$$

$$c(i,T) = \sqrt{\frac{\log(T)}{n(i,T)}}$$

intuitions:

- if $n(i,T)$ is small (little information) then $c(i,T)$ is large : we need to explore
- if $n(i,T)$ is large then $c(i,T)$ is small, except when T grows, but exponentially less often

Why $c(i,T) = \sqrt{\frac{\log(T)}{n(i,T)}}$?

Chernoff-Hoeffding bounds

Let $y_1, \dots, y_n$ iid samples of Y

Law of large numbers: $\frac{1}{n}\sum_{i=1}^{n} y_i \longrightarrow \mathbb{E}[Y]$

Better: this happens fast!

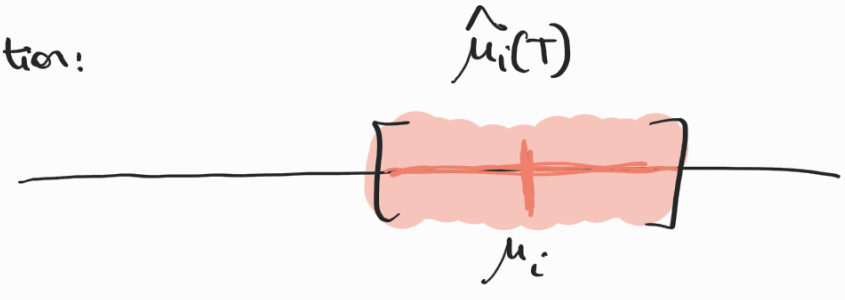
$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} y_i - \mathbb{E}[Y]\right| \geq c\right) \leq 2e^{-2c^2 n}$$

We apply it to our case:

$$\mathbb{P}\left(\left|\hat{\mu}_i(T) - \mu_i\right| \geq c(i,T)\right) \leq 2\, e^{-2\,c(i,T)^2\, n(i,T)}$$

$c(i,T) = \sqrt{\frac{\log(T)}{n(i,T)}}$ gives $\frac{2}{T^2} \xrightarrow[T\to+\infty]{} 0$

Intuition:



with high probability

UPDATES

$$\hat{\mu}_i(T) = \frac{1}{n(i,T)} \sum_{t=1}^{T} r(i,t)$$

After choosing i :

$$\hat{\mu}_i(T+1) = \frac{1}{\underbrace{n(i,T+1)}_{n(i,T)+1}} \sum_{t=1}^{T+1} r(i,t)$$

Small calculations

$$\hat{\mu}_i(T+1) = \hat{\mu}_i(T) + \frac{1}{n(i,T)+1}\left[X(i,T+1) - \hat{\mu}_i(T)\right]$$

$$\text{NEW} = \text{OLD} + \alpha\left[\text{CURRENT} - \text{OLD}\right]$$

We call this **step size**

Two possible updates :

- empirical mean
- constant step size ($\alpha = 0.1$ for instance)

$$\hat{\mu}_i(T+1) = \hat{\mu}_i(T) + \alpha\left(\hat{\mu}_i(T) - X(i,T+1)\right)$$

THOMPSON SAMPLING ≡ POSTERIOR SAMPLING

Key idea: instead of choosing the best action from the **model**, look at the model as a **distribution** of which action is best

Bernoulli bandit

θ_i : probability of success for arm i

$\alpha(i,t)$: number of successes for i } up to time t

$\beta(i,t)$: number of failures for i } up to time t

Greedy

$$\hat{\theta}(i,T) = \frac{\alpha(i,T)}{\alpha(i,T) + \beta(i,T)}$$

Thompson

$$\hat{\theta}(i,T) = \text{Sample}\left(\alpha(i,t), \beta(i,t)\right)$$

Choose $\underset{i \in [1,K]}{\operatorname{argmax}} \hat{\theta}(i,T)$

Update $\alpha(i,t), \beta(i,t)$

Update:

$$\alpha(i,t+1), \beta(i,t+1) =$$

$$\begin{cases} \alpha(i,t), \beta(i,t) & \text{if } i \text{ not chosen} \\ (\alpha(i,t), \beta(i,t)) + (r(t), 1-r(t)) & \text{o/w} \end{cases}$$

$$r(t) = \begin{cases} 1 & \text{if success} \\ 0 & \text{if failure} \end{cases}$$

β distribution:

$$P(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

Here Bayesian inference is easy (computationally). In general it's hard → approximation efforts