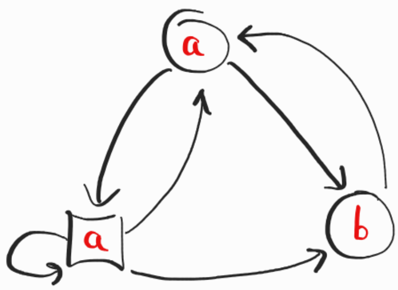
PREVIOUSLY

Eve ○ σ

Adam □ τ



Theorem Let $\mathcal{G}$ a reachability game

$\exists \sigma$ positional strategy winning from $W_{Eve}(\mathcal{G})$

$\exists \tau$ ———————————————— $W_{Adam}(\mathcal{G}) = V \setminus W_{Eve}(\mathcal{G})$

$W_{Eve}(\mathcal{G})$, σ, and τ, can be computed in $\underbrace{O(n+m)}_{\text{linear time}} \approx O(m)$

Theorem Let $\mathcal{G}$ a Büchi game

$\exists \sigma$ positional strategy winning from $W_{Eve}(\mathcal{G})$

$\exists \tau$ ———————————————— $W_{Adam}(\mathcal{G}) = V \setminus W_{Eve}(\mathcal{G})$

$W_{Eve}(\mathcal{G})$, σ, and τ, can be computed in $O(n\, m)$

quadratic

Positional Determinacy: Reachability

$$\mathrm{Pre}_{Eve}(X) = \{v \in V_{Eve} : \exists (v,v') \in E \quad v' \in X\} \cup \{v \in V_{Adam} : \forall (v,v') \in E \quad v' \in X\}$$

Algorithm: $\begin{cases} X_0 = \emptyset \\ X_{k+1} = F \cup \mathrm{Pre}_{Eve}(X_k) \end{cases}$

$$X_0 \subseteq X_1 \subseteq \dots \subseteq X_k = W_E(\mathcal{G})$$

For Eve

For $v \in W_E(\mathcal{G})$ define $\mathrm{rank}(v) = \min\{p : v \in X_p\}$:

→ If $v \in V_{Eve}$ $\mathrm{rank}(v) = p+1 \Rightarrow \exists (v,v') \in E$ $\mathrm{rank}(v') \leq p$ or $v \in F$

↳ Define $\sigma(v) = (v,v')$

→ If $v \in V_{Adam}$ $\mathrm{rank}(v) = p+1 \Rightarrow \forall (v,v') \in E$ $\mathrm{rank}(v') \leq p$ or $v \in F$

Claim: σ positional winning from $W_{Eve}(\mathcal{G})$

→ along a play rank strictly decreases until reaching F

For Adam:

$$F \cup Pre_{Eve}(W_E(\mathcal{G})) \subseteq W_E(\mathcal{G})$$

$$\Rightarrow \quad W_A(\mathcal{G}) \subseteq V \setminus \left(F \cup Pre_{Eve}(W_E(\mathcal{G}))\right)$$

For $v \in W_A(\mathcal{G})$: first, $v \notin F$ and:

→ If $v \in V_{Adam}$ $\quad \Rightarrow \exists (v,v') \in E \quad v' \notin W_E(\mathcal{G})$

↳ Define $\tau(v) = (v, v')$

→ If $v \in V_{Eve}$ $\quad \Rightarrow \forall (v,v') \in E \quad v' \notin W_E(\mathcal{G})$

Claim: τ positional winning from $W_A(\mathcal{G})$

→ along a play remains in $W_A(\mathcal{G})$
which does not intersect F

Positional Determinacy: Büchi

$$\begin{cases} Y_0 = V \\ Y_{k+1} = \mathrm{Attr}_E(F \cap \mathrm{Pre}_E(Y_k)) \end{cases}$$

$$Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_k = W_E(G)$$

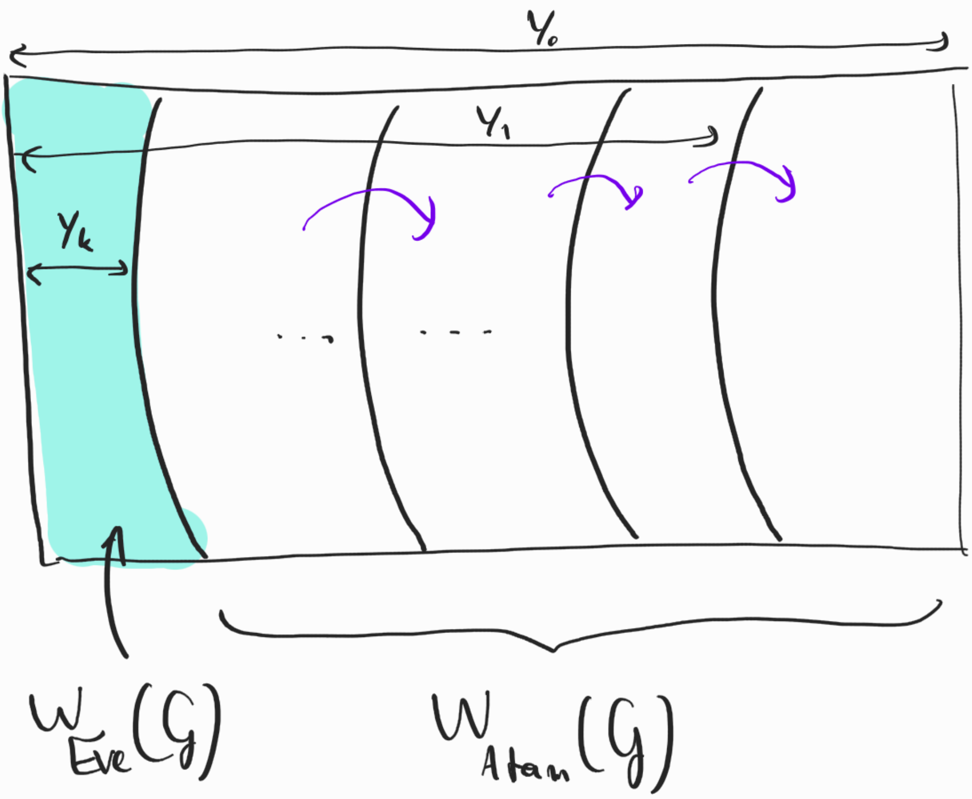
For Eve

$$W_E(G) \subseteq \mathrm{Attr}_E(F \cap \mathrm{Pre}_E(W_E(G)))$$

$$\sigma(v) = \begin{cases} \sigma_{\mathrm{Attr}_E}(v) & \text{if } v \notin F \cap \mathrm{Pre}_E(W_E(G)) \\ (v,v') \text{ with } v' \in W_E(G) & \text{if } v \in F \cap \mathrm{Pre}_{Eve}(W_E(G)) \end{cases}$$

Claim: σ positional winning from $W_E(G)$

For Adam:



$$Y_{k+1} = \mathrm{Attr}_{Eve}\left(F \cap \mathrm{Pre}_{Eve}(Y_k)\right)$$

counter attractor strategy

τ_{k+1} ensures from $V \setminus Y_{k+1}$ never to reach $F \cap \mathrm{Pre}_{Eve}(Y_k)$ and to stay in $V \setminus Y_{k+1}$

For $v \in W_A(G)$ define $\mathrm{rank}(v) = \min\{k : v \notin Y_k\}$:

$\mathrm{rank}(v) = k+1 \quad : \quad v \in Y_k \setminus Y_{k+1}$

→ If $v \in V_{Adam}$ $\mathrm{rank}(v) = k+1$: two cases:

- $v \in F$: then $v \notin \mathrm{Pre}_{Eve}(Y_k)$ so
  $\exists (v,v') \in E \quad v' \notin Y_k$ ie $\mathrm{rank}(v') \leq k$
  ↳ define $\tau(v) = (v, v')$

- $v \notin F$: $v \in V \setminus Y_{k+1}$ play $\tau_k(v) = (v, v')$
  since it ensures to stay in $V \setminus Y_{k+1}$, $\mathrm{rank}(v') \leq k+1$

→ If $v \in V_{Eve}$ $\mathrm{rank}(v) = k+1$: two cases :

- $v \in F$: then $v \notin \mathrm{Pre}_{Eve}(Y_k)$ so
  $\forall (v,v') \in E \quad v' \notin Y_k$ ie $\mathrm{rank}(v') \leq k$

- $v \notin F$: $v \in V \setminus Y_{k+1}$
  $\forall (v,v') \in E \quad v' \notin Y_{k+1}$ ie $\mathrm{rank}(v') \leq k+1$

Claim: τ positional winning from $W_A(\mathcal{G})$

τ ensures that | the rank never increases
| when visiting F, the rank decreases

So a play consistent with τ sees finitely many F.