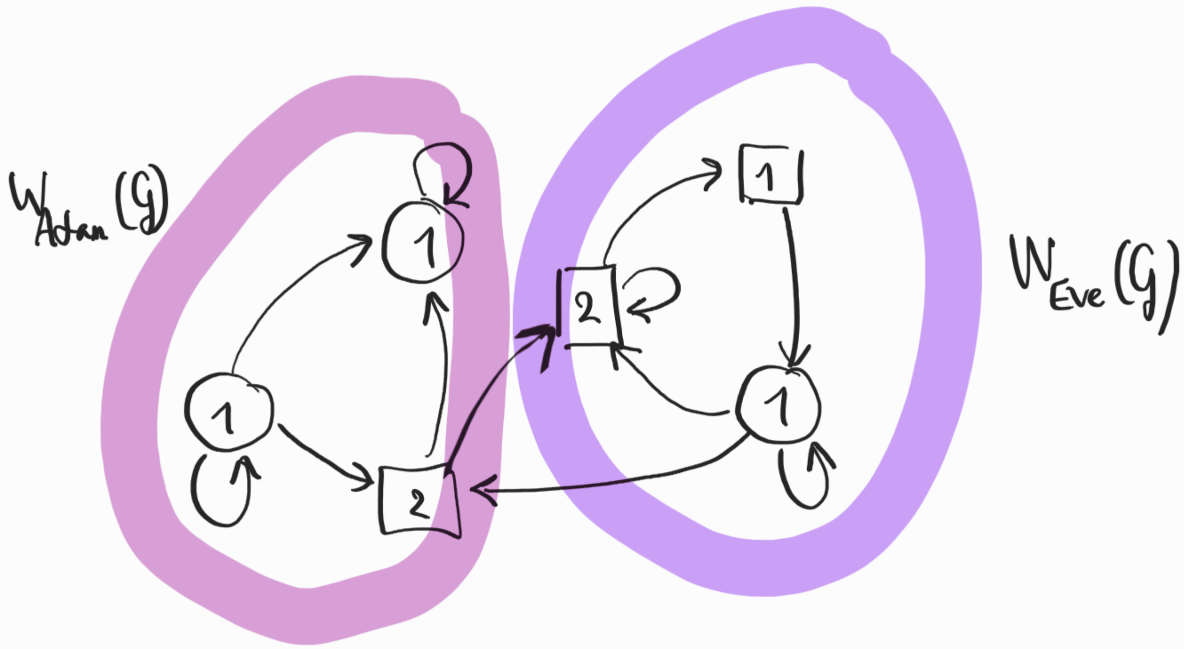
BÜCHI / CoBÜCHI

Definitions $C = \{1, 2\}$

$$\text{BÜCHI} = \{ \rho \in \{1,2\}^\omega : \ \forall i \ \exists j \geq i \quad \rho_j = 2 \}$$

$$\text{CoBÜCHI} = \{ \rho \in \{1,2\}^\omega : \ \exists i \ \forall j \geq i \quad \rho_j = 1 \}$$



Convention

$F = col^{-1}(\{2\})$

BÜCHI(F) : see F infinitely many times

Theorem Let $\mathcal{G}$ a Büchi game

$\exists \sigma$ positional strategy winning from $W_{Eve}(\mathcal{G})$

$\exists \tau$ ——————————— $W_{Adam}(\mathcal{G}) = V \setminus W_{Eve}(\mathcal{G})$

$W_{Eve}(\mathcal{G})$, σ, and τ, can be computed in $O(n\,m)$

quadratic

Define

$$\phi : \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$$

$$X \longmapsto Attr_{Eve}(F \cap Pre_{Eve}(X))$$

ϕ is monotonic : $X \subseteq Y \Rightarrow \phi(X) \subseteq \phi(Y)$

Theorem (Knaster-Tarski, dual version)

$(\mathcal{L}, \leq)$ complete finite lattice

$\phi : (\mathcal{L}, \leq) \longrightarrow (\mathcal{L}, \leq)$ monotonic

(1) ϕ has a unique greatest fix point: $\phi(x) = x$

(2) the greatest fixpoint is also the greatest postfixpoint: $\phi(x) \geq x$

(3) they are computed by: $\begin{cases} x_0 = \top \\ x_{k+1} = \phi(x_k) \end{cases}$

Lemma $\quad F = col^{-1}(\{2\})$

$W_{Eve}(G)$ is the greatest fixpoint of ϕ

post

Proof (1) $W_{Eve}(\mathcal{G})$ is a postfixpoint of Φ:

$$\mathrm{Attr}_{Eve}(F \cap \mathrm{Pre}_{Eve}(W_{Eve}(\mathcal{G}))) \supseteq W_{Eve}(\mathcal{G})$$

(2) $W_{Eve}(\mathcal{G})$ contains all fixpoints of Φ:

$$\mathrm{Attr}_{Eve}(F \cap \mathrm{Pre}_{Eve}(X)) = X \Rightarrow X \subseteq W_{Eve}(\mathcal{G})$$

(1) We prove

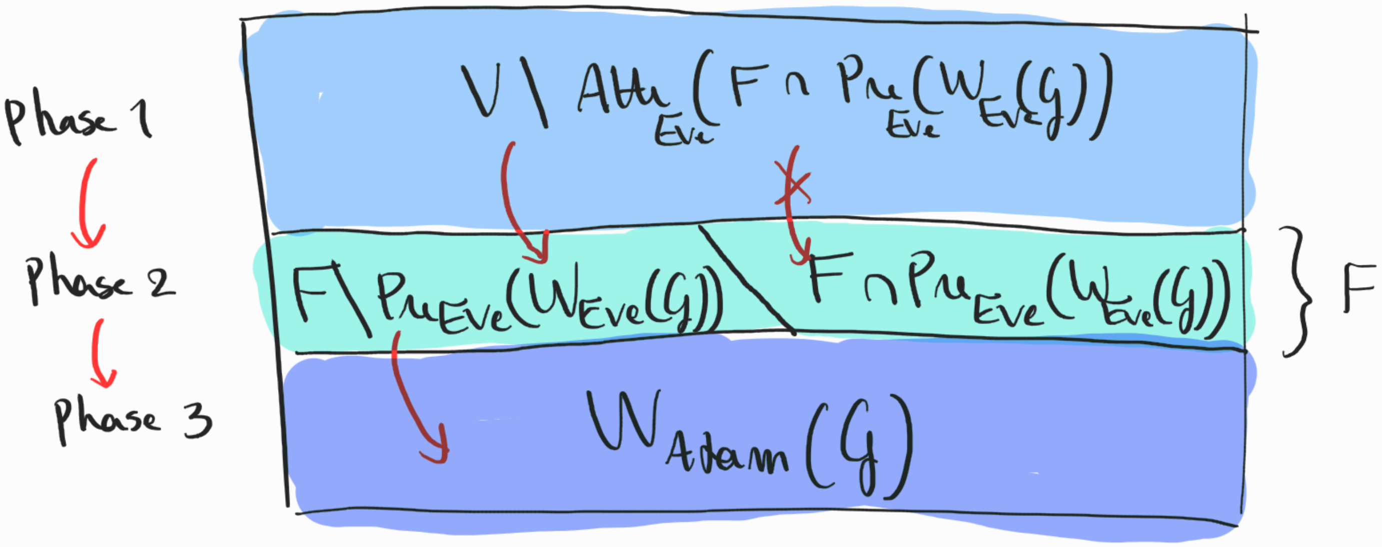
$$X = V \setminus \mathrm{Attr}_{Eve}(F \cap \mathrm{Pre}_{Eve}(W_{Eve}(\mathcal{G}))) \subseteq W_{Adam}(\mathcal{G})$$

let:

- τ_{Attr} ensuring never $F \cap \mathrm{Pre}_{Eve}(W_{Eve}(\mathcal{G}))$ from X
- τ_{win} winning from $W_{Adam}(\mathcal{G})$

For $v \in X \cup W_{Adam}(\mathcal{G})$, define:

$$\tau(v) = \begin{cases} \tau_{Attr}(v) & \text{if } v \in X \setminus F \\ (v, v') \text{ with } v' \in W_{Adam}(\mathcal{G}) & \text{if } v \in F \setminus \mathrm{Pre}_{Eve}(W_{Eve}(\mathcal{G})) \\ \tau_{win}(v) & \text{if } v \in W_{Adam}(\mathcal{G}) \end{cases}$$



We claim that τ ensures CoBüchi(F).

Consider a play consistent with τ:

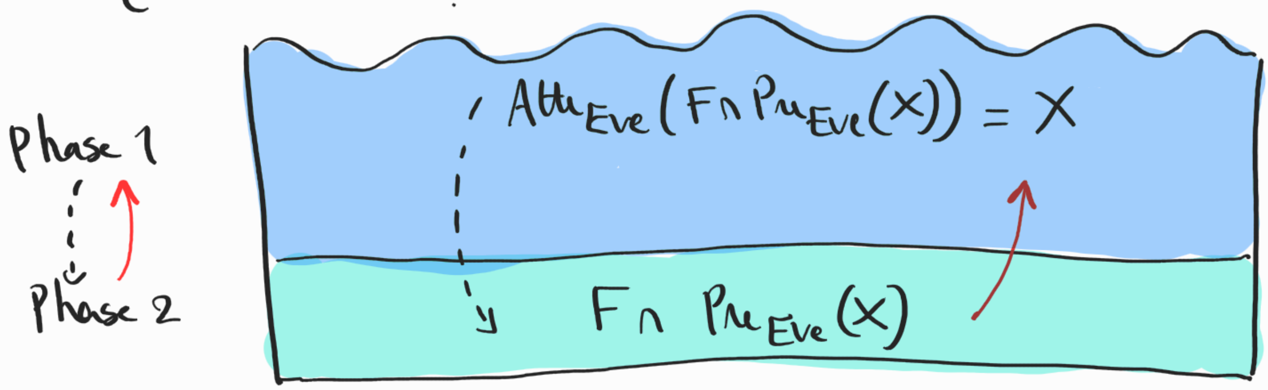
- either never reaches F, ok.
- or reaches F, then $F \setminus \mathrm{Pre}_{\mathrm{Eve}}(W_{\mathrm{Eve}}(G))$

  so next vertex is in $W_{\mathrm{Adam}}(G)$,
  from where CoBüchi(F) is satisfied

(2) let X such that $\mathrm{Attr}_{Eve}(F \cap \mathrm{Pre}_{Eve}(X)) = X$
we show that $X \subseteq W_{Eve}(G)$

let σ_{Attr} ensuring $F \cap \mathrm{Pre}_{Eve}(X)$ from $\mathrm{Attr}_{Eve}(F \cap \mathrm{Pre}_{Eve}(X))$

$$\sigma(v) = \begin{cases} \sigma_{Attr}(v) & \text{on } \mathrm{Attr}_{Eve}(F \cap \mathrm{Pre}_{Eve}(X)) \setminus F \cap \mathrm{Pre}_{Eve}(X) \\ (v, v') \text{ with } v' \in X & \text{if } v \in F \cap \mathrm{Pre}_{Eve}(X) \end{cases}$$



We claim that σ ensures $\mathrm{Büchi}(F)$ from X.

For any fixed point X of Φ we constructed a positional winning strategy from X

$\square$

Algorithm

$X = V$

while $(\Phi(X) \neq X)$

$\quad X \leftarrow \Phi(X)$

Return X

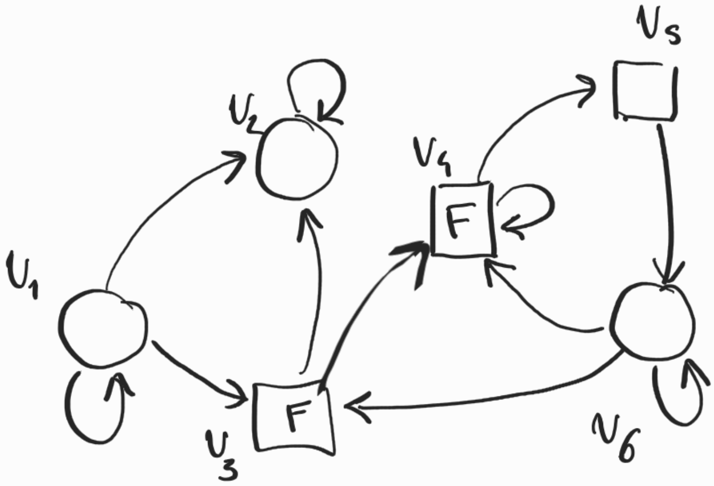
$O(n + m)$

complexity:

$O(n$

at most n iterations of while loop: decreasing sequence of subsets of vertices

cost of computing Φ



$$X_0 = V$$

$$X_1 = Attr_{Eve}(F \cap Pre_{Eve}(X_0)) = V \setminus \{v_2\}$$

$$X_2 = Attr_{Eve}(F \cap Pre_{Eve}(X_1)) = \{v_4, v_5, v_6\}$$

$$X_3 = X_2$$

Corollary: positional determinacy

For Eve:

The construction of σ in the proof of the lemma yields σ positional winning from $W_{Eve}(G)$

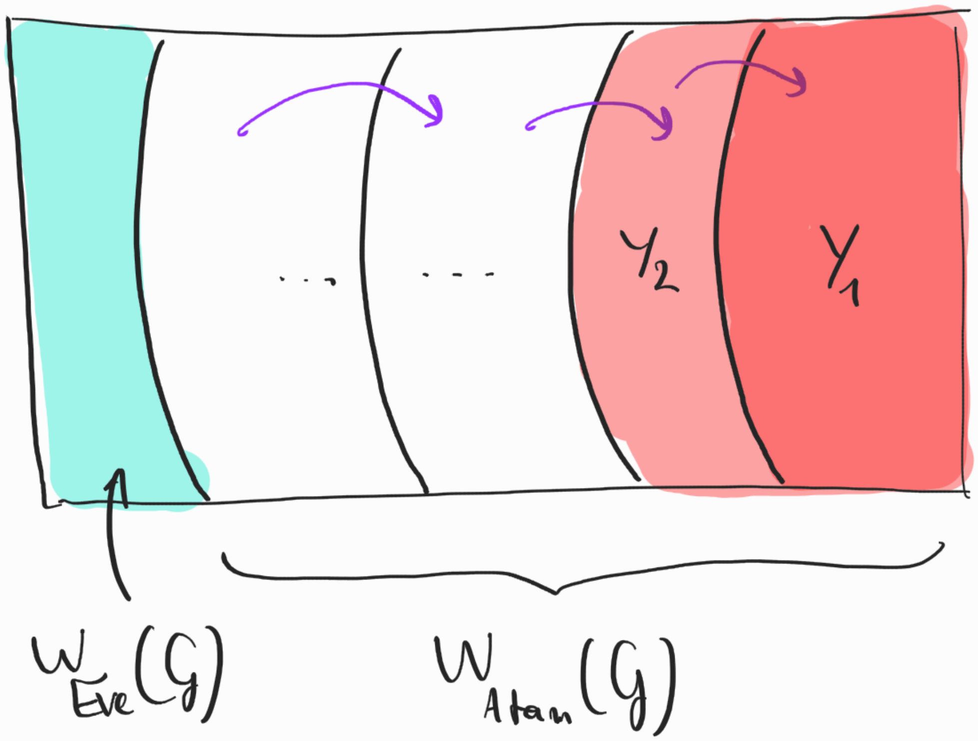
For Adam:

$X_0 = V$ $\qquad$ $Y_0 = V \setminus X_0 = \emptyset$

$X_{k+1} = Attr_{Eve}(F \cap Pre_{Eve}(X_k))$ $\qquad$ $Y_{k+1} = V \setminus X_k$

$$X_0 \supseteq X_1 \supseteq \dots \supseteq X_n = W_{Eve}(G)$$

$$Y_0 \subseteq Y_1 \subseteq \dots \subseteq Y_n = W_{Adam}(G)$$



$$Y_{k+1} = V \setminus Attr_{Eve}\left(F \cap Pre_{Eve}(V \setminus Y_k)\right)$$

counter-attractor strategy

τ_{k+1} ensures from Y_{k+1} never to reach $F \cap Pre_{Eve}(V \setminus Y_k)$

$$\tau(v) = \begin{cases} \tau_{k+1}(v) & \text{if } v \in Y_{k+1} \setminus Y_k \setminus F \\ (v, v') \text{ with } v' \in Y_k & \text{if } v \in F \setminus Pre_{Eve}(V \setminus Y_k) \end{cases}$$

τ ensures that when visiting F, we go from Y_{k+1} to Y_k.

So a play consistent with τ sees finitely many F.

τ is positional

$\square$