REACHABILITY/SAFETY

Definitions $C = \{0,1\}$

Dual

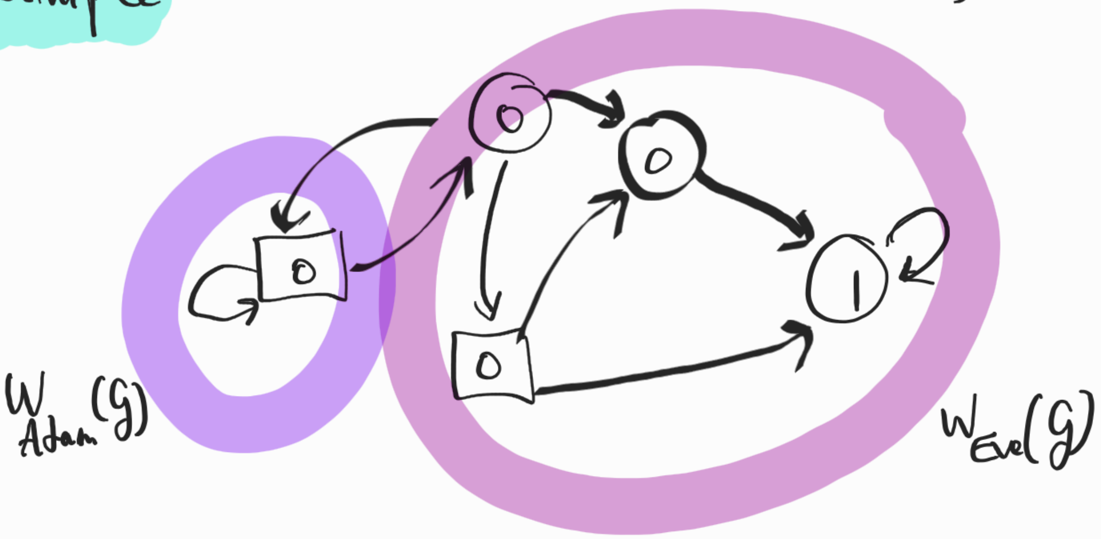
$$\mathrm{REACH} = \{ \rho \in \{0,1\}^\omega : \exists i \ \rho_i = 1 \}$$

$$\mathrm{SAFE} = \{ \rho \in \{0,1\}^\omega : \forall i \ \rho_i = 0 \} = \{0,1\}^\omega \setminus \mathrm{REACH}$$

If Eve has reachability objective, Adam has safety

Example

$F = col^{-1}(\{1\}) \subseteq V$



Theorem Let $\mathcal{G}$ a reachability game

$\exists\, \sigma$ positional strategy winning from $W_{Eve}(\mathcal{G})$

$\exists\, \tau$ ——————————— $W_{Adam}(\mathcal{G}) = V \setminus W_{Eve}(\mathcal{G})$

$W_{Eve}(\mathcal{G})$, σ, and τ, can be computed in $\underbrace{O(n+m) = O(m)}_{\text{linear time}}$

Consider $\quad Pre_{Eve} : \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$

$$Pre_{Eve}(X) = \{ v \in V_{Eve} : \exists (v,v') \in E,\ v' \in X \} \cup \{ v \in V_{Adam} : \forall (v,v') \in E,\ v' \in X \} \qquad \subseteq V$$

Facts

- $(\mathcal{P}(V), \subseteq)$ complete lattice
- $\mathrm{Pre}_{\mathrm{Eve}}$ monotonic : $X \subseteq Y \Rightarrow \mathrm{Pre}_{\mathrm{Eve}}(X) \subseteq \mathrm{Pre}_{\mathrm{Eve}}(Y)$

we can remove finite and add "preserves infima"

Theorem (Knaster-Tarski)

$(\mathcal{L}, \leq)$ complete finite lattice

$\phi : (\mathcal{L}, \leq) \longrightarrow (\mathcal{L}, \leq)$ monotonic

$x \leq y \Rightarrow \phi(x) \leq \phi(y)$

(1) ϕ has a unique least fixpoint: $\phi(x) = x$

(2) the least fixpoint is also the least prefixpoint : $\phi(x) \leq x$

(3) they are computed by : $\begin{cases} x_0 = \bot \\ x_{k+1} = \phi(x_k) \end{cases}$

$x_0 \leq x_1 \leq x_2 \leq \ldots \quad x_k = x_{k+1}$

x_k is the least (pre) fixpoint

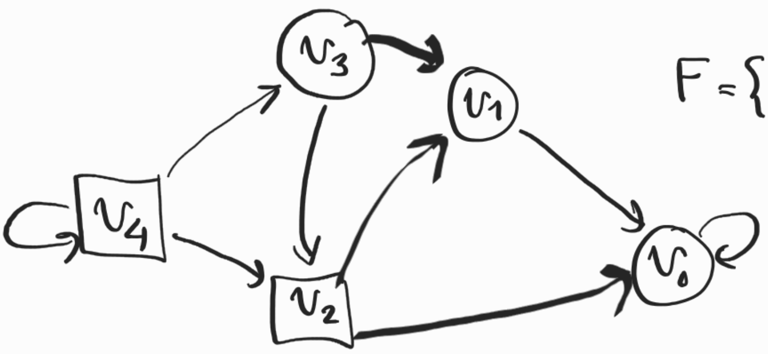
Lemma $\mathcal{G}$ reachability game $\mathcal{G} = (\mathcal{A}, \mathrm{Reach}(F))$

$W_{Eve}(\mathcal{G})$ is the least fixpoint of $\Phi : X \mapsto F \cup \mathrm{Pre}_{Eve}(X)$

pre

Often called the attractor:

We write $W_{Eve}(\mathcal{G}) = \mathrm{Attr}_{Eve}(F)$



$F = \{v_0\}$

$X_0 = \emptyset$

$X_1 = F \cup \mathrm{Pre}_{Eve}(X_0) = \{v_0\}$

$X_2 = F \cup \mathrm{Pre}_{Eve}(X_1) = \{v_0, v_1\}$

$X_3 = \{v_0, v_1, v_3, v_2\}$

$X_4 = X_3$

Proof (1) $W_E(\mathcal{G})$ is a prefixpoint of Φ

$$F \cup \mathrm{Pre}_{Eve}(W_E(\mathcal{G})) \subseteq W_E(\mathcal{G})$$

(2) $W_E(\mathcal{G})$ is included in any prefixpoint of Φ

$$F \cup \mathrm{Pre}_{Eve}(X) \subseteq X \implies W_{Eve}(\mathcal{G}) \subseteq X$$

(1) $F \cup \mathrm{Pre}_{Eve}(W_{Eve}(\mathcal{G})) \subseteq W_{Eve}(\mathcal{G})$

We construct a winning strategy from $\mathrm{Pre}_{Eve}(W_{Eve}(\mathcal{G}))$

- First step: ensure to reach $W_{Eve}(\mathcal{G})$
- After that: play a winning strategy

(2) We prove (considering the other player)

$$V \setminus X \subseteq V \setminus (F \cup \mathrm{Pre}_{Eve}(X)) \implies V \setminus X \subseteq W_{Adam}(\mathcal{G})$$

Let $v \in V \setminus X$, define

$\tau(v) = (v, v')$ such that $v' \in V \setminus X$

τ ensures from $V \setminus X$ to stay in $V \setminus X$:
it is winning because $V \setminus X \subseteq V \setminus F$

For any prefixpoint X of Pre_{Eve}, we constructed a positional winning strategy from $V \setminus X$ $\square$

Algorithm (Knaster-Tarski instantiated)

$X = \emptyset$
while ($\Phi(X) \neq X$) :
    $X \leftarrow \Phi(X)$

Return X

(1) at most n iterations :
inclusion increasing subsets of vertices

(2) computing $\mathrm{FuPre}_{\mathrm{Eve}}(X)$ can be done in $O(m)$

Naive complexity :

$O(n\,m)$

Corollary: positional determinacy

For Eve: $\begin{cases} X_0 = \emptyset \\ X_{k+1} = F \cup Pre_{Eve}(X_k) \end{cases}$

$$X_0 \subseteq X_1 \subseteq \dots \subseteq X_k = W_E(\mathcal{G})$$

For $v \in W_E(\mathcal{G})$ define $rank(v) = \min\{p : v \in X_p\}$

For $v \in V_{Eve}$ $rank(v) = p+1 \Rightarrow \exists (v,v') \in E$ $rank(v') \leq p$
or $v' \in F$
↳ Define $\sigma(v) = (v,v')$

Claim: σ positional winning from $W_{Eve}(\mathcal{G})$
→ along a play rank strictly decreases
until reaching F

For Adam: We apply the construction of τ in the proof of the lemma to $X = V \setminus W_{Eve}(\mathcal{G})$: this yields τ positional winning from X, so $X = W_{Adam}(\mathcal{G})$ $\square$