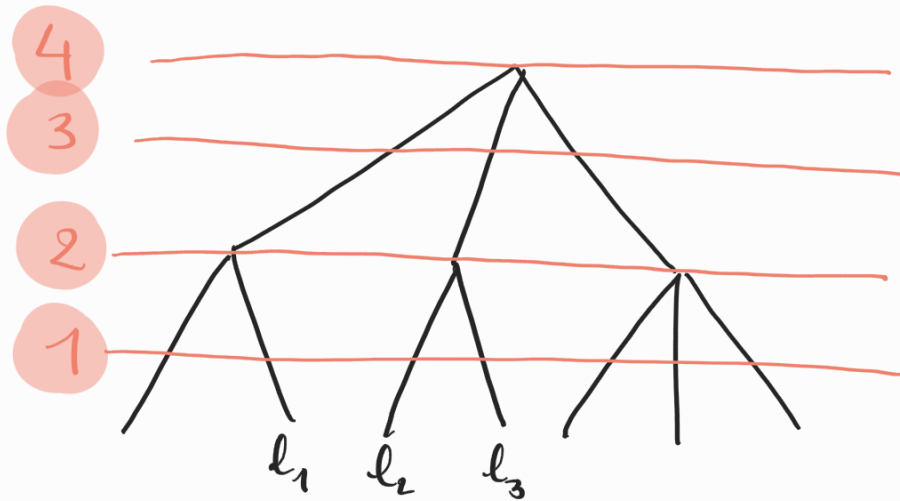


# QUASIPOLYNOMIAL VALUE ITERATION

G parity game with  $n$  vertices over  $[1, d]$



Examples:

$$l_1 <_1 l_2 <_1 l_3$$

$$l_1 <_2 l_2 \equiv_2 l_3$$

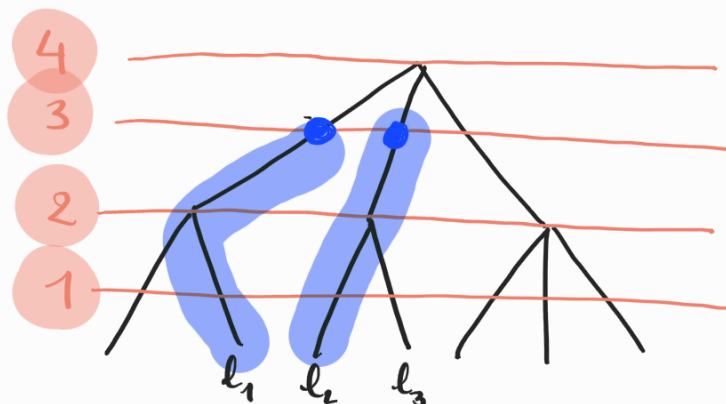
$$l_1 <_3 l_2 \equiv_3 l_3$$

$$l_1 \equiv_4 l_2 \equiv_4 l_3$$

A tree of height  $d/2$  defines  $d$  orders on leaves:

$\left| \begin{array}{l} p \text{ even (node levels)} \\ p \text{ odd (intermediate levels)} \end{array} \right.$

$l <_p l'$  if the ancestor of  $l$  at level  $p$  is to the left of the ancestor of  $l'$  at level  $p$



$$l_1 <_3 l_2$$

**Definition:**  $t$  a  $(n, 1/2)$ -tree (identified with leaves of  $t$ )

$$\mu: V \rightarrow t \cup \{T\}$$

$(v, v') \in E$  is progressive if

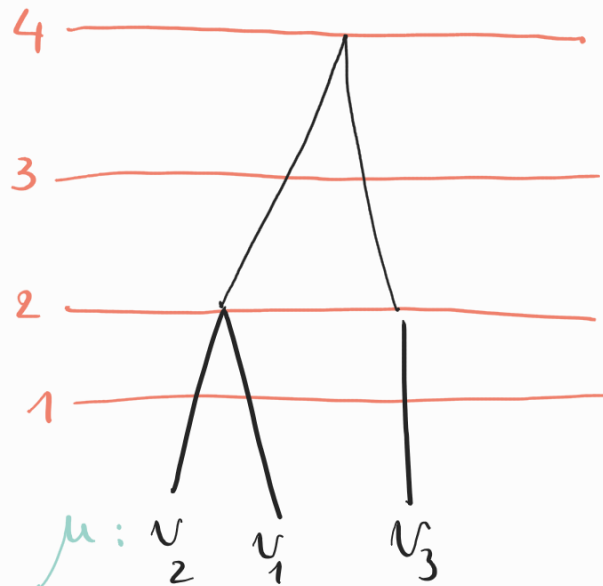
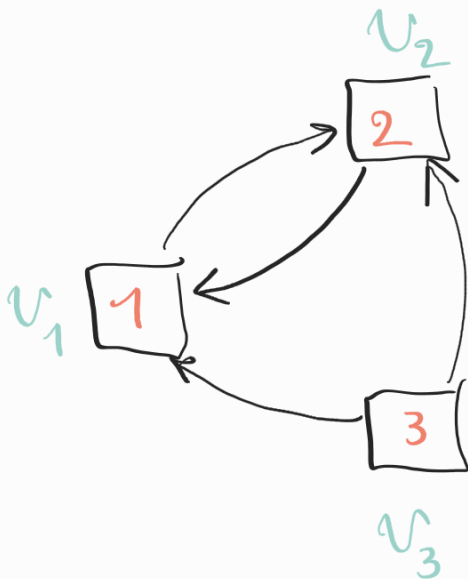
$$t <_p T \quad \forall p$$

$$\text{col}(v)=p \quad \left| \begin{array}{l} p \text{ even} : \mu(v) \geq_p \mu(v') \\ p \text{ odd} : \mu(v) >_p \mu(v') \end{array} \right. \quad \begin{array}{l} \text{non-strict} \\ \text{strict} \end{array}$$

$\mu$  is a progress measure if

$$\forall v \in V_{\text{Eve}} \quad \exists (v, v') \in E \text{ progressive}$$

$$\forall v \in V_{\text{Adam}} \quad \forall (v, v') \in E \text{ progressive}$$



$$\begin{aligned} \mu(v_3) >_3 \mu(v_1) \wedge \mu(v_3) >_3 \mu(v_2) \\ \mu(v_1) >_1 \mu(v_2) \\ \mu(v_2) >_2 \mu(v_1) \end{aligned}$$

## The fundamental theorem:

- (1)  $\forall t$  a  $(n, d/2)$ -tree,  $\forall \mu: V \rightarrow t \cup \{T\}$  progress measure  
 $\forall v \quad \mu(v) \neq T \Rightarrow v \in W_{\text{Eve}}(G)$
- (2)  $\exists t$  a  $(n, d/2)$ -tree,  $\exists \mu: V \rightarrow t \cup \{T\}$  progress measure  
 $\forall v \quad \mu(v) \neq T \Leftarrow v \in W_{\text{Eve}}(G)$

Theorem:  $G$  graph over  $[1, d]$

- (1)  $\forall t$  a  $(n, d/2)$ -tree,  $\forall \mu: V \rightarrow t \cup \{T\}$  progress measure  
 $\forall v \quad \mu(v) \neq T \Rightarrow G, v \models \text{Parity}$
- (2)  $\exists t$  a  $(n, d/2)$ -tree,  $\exists \mu: V \rightarrow t \cup \{T\}$  progress measure  
 $\forall v \quad \mu(v) \neq T \Leftarrow G, v \models \text{Parity}$

Proof:

(1) let  $\mu$  progress measure for  $G$  and  $\mu(v_0) \neq T$

\* for all  $v$  reachable from  $v_0$ ,  $\mu(v) \neq T$

$$v_0 \xrightarrow{p_0} v_1 \xrightarrow{p_1} v_2 \dots v_k \xrightarrow{p_k} v$$

$$\mu(v_0) \geq_{p_0} \mu(v_1) \geq_{p_1} \dots \geq_{p_k} \mu(v)$$

\* all cycles reachable from  $v$  are even

$$v_1 \xrightarrow{p_1} v_2 \xrightarrow{p_2} v_3 \xrightarrow{p_3} \dots v_k \xrightarrow{p_k} v_1$$

Towards contradiction: odd cycle,  
 $p_1$  maximum odd priority

could be  $<$  or  $\leq$

$$\mu(v_1) >_{p_1} \mu(v_2) \geq_{p_2} \dots \geq \mu(v_k) \geq_{p_k} \mu(v_1)$$

Fact:  $q \leq p$  and  $x \leq_q x' \Rightarrow x \leq_p x'$

So:

$$\mu(v_1) >_{p_1} \mu(v_2) \geq_{p_1} \dots \geq \mu(v_k) \geq_{p_1} \mu(v_1)$$

Fact:  $x <_p x' \leq_p x'' \Rightarrow x <_p x''$

So  $\mu(v_1) >_{p_1} \mu(v_1)$ : contradiction

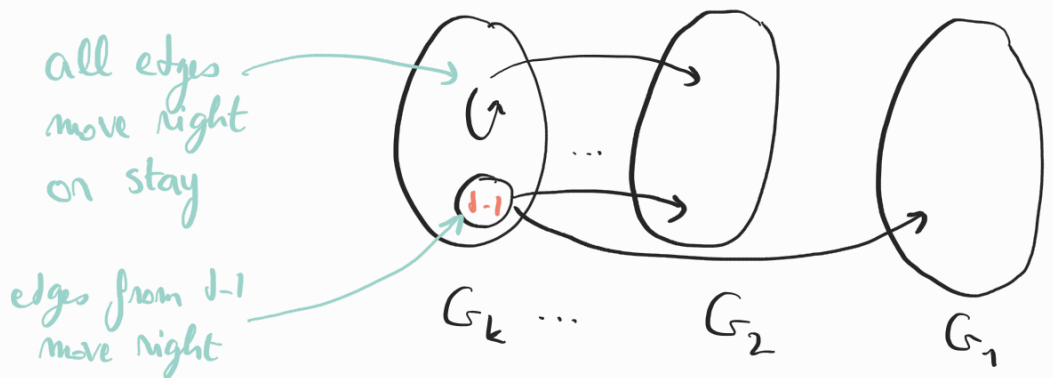


(2) By induction on  $d$

Assume  $G \models \text{Parity}$  ( $T$  to the other vertices)

$G'$  :  $G$  where we remove outgoing edges to  $\{v \in V : \text{col}(v) = d\}$

We consider a decomposition of  $G'$  into strongly connected components:



priorities  $d-1$  have no outgoing edges

Each  $G_i \models \text{Parity}[1, d-2]$

so  $\exists t_i \exists \mu_i : V(G_i) \rightarrow t_i$

Define  $t$  :



$\mu$  :



Claim:  $\mu$  is a progress measure for  $G$   
let  $(v, v') \in E$

case 1:  $\text{col}(v) = d$

then  $\mu(v) \geq_d \mu(v')$

because  $x \leq_d y$  always holds!

case 2:  $\text{col}(v) = d-1$

$v \in G_i \Rightarrow v' \in G_j \quad j > i$

$\mu(v) >_{d-1} \mu(v')$

case 3:  $\text{col}(v) = p < d-1$

either  $v, v' \in G_i$  : induction hypothesis on  $\mu_i$

or  $v \in G_i$  and  $v' \in G_j \quad j > i$

then  $\mu(v) >_p \mu(v')$

□

## Proof: of the fundamental theorem

(1) Define  $\sigma(v) = (v, v')$  some progressive edge.

$G|_{\sigma}$  : graph restricted to moves of  $\sigma$

$\mu$  is a progress measure for  $G|_{\sigma}$

so  $G|_{\sigma}, v \models \text{Parity}$

so  $\sigma$  is winning from  $v$

(2) We use positional determinacy **HERE**

let  $\sigma$  winning positional strategy

Consider  $G|_{\sigma}$ .

For  $v \in W_{\text{Eve}}(G)$ , we have  $G|_{\sigma}, v \models \text{Parity}$ .

Thanks to the theorem above for graphs,  
there exists  $\mu$  progress measure for  $G|_{\sigma}$  such that:

$$G|_{\sigma}, v \models \text{Parity} \Rightarrow \mu(v) \neq \perp$$

Now  $\mu$  is also a progress measure for  $G$  such that:

$$v \in W_{\text{Eve}}(G) \Rightarrow \mu(v) \neq \perp$$

□

Fix  $\mathcal{L}$  a  $(n, 1/2)$ -universal tree

$\mathcal{L}$  is the set of functions  $\mu: V \rightarrow \mathcal{L} \cup \{\top\}$   
equipped with componentwise comparison:

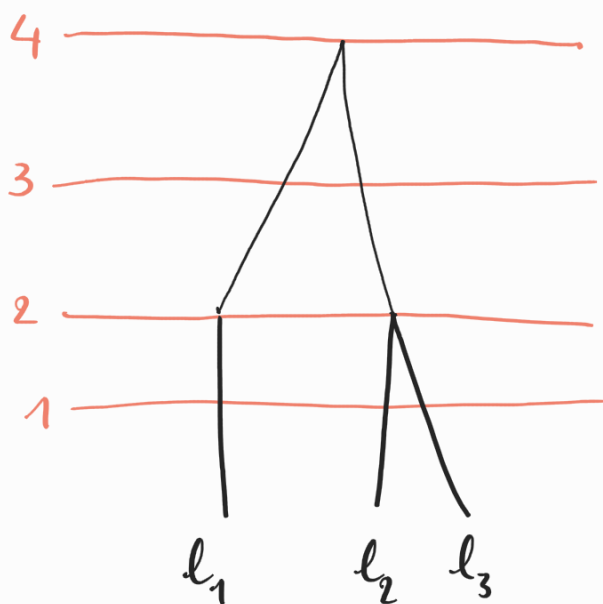
$$\mu \leq \mu' \quad \text{if} \quad \forall v \quad \mu(v) \leq_1 \mu'(v)$$

$\mathcal{L}$  is a complete lattice

Define  $\delta: \mathcal{L} \times [1, d] \rightarrow \mathcal{L} \cup \{\top\}$

$$p \text{ even: } \delta(l, p) = \min \{ l' : l \leq_p l' \}$$

$$p \text{ odd: } \delta(l, p) = \min \{ l' : l <_p l' \}$$



$$\delta(l_1, 1) = l_2$$

$$\delta(l_2, 1) = l_3$$

$$\delta(l_3, 2) = l_2$$

$$\delta(l_2, 4) = l_1$$

$$\delta(l_3, 1) = \top$$

Fact:  $\text{col}(v) = p$

$$l \geq \delta(l', p) \Leftrightarrow \begin{cases} p \text{ even} : & l \geq_p l' \\ p \text{ odd} : & l >_p l' \end{cases}$$

Define Update:  $\mathcal{L} \rightarrow \mathcal{L}$

$$\text{Update}(\mu)(v) = \begin{cases} \text{if } v \in V_{\text{Even}} & \min_{(v, v') \in E} \delta(\mu(v'), \text{col}(v)) \\ \text{if } v \in V_{\text{Odd}} & \max_{(v, v') \in E} \delta(\mu(v'), \text{col}(v)) \end{cases}$$

Lemma: progress measure  $\Leftrightarrow$  prefixpoint of Update

Proof  $\mu \geq \text{Update}(\mu)$

$$\Leftrightarrow \begin{cases} v \in V_{\text{Even}} & \mu(v) \geq \min \{ \delta(\mu(v'), \text{col}(v)) : (v, v') \in E \} \\ v \in V_{\text{Odd}} & \mu(v) \geq \max \{ \delta(\mu(v'), \text{col}(v)) : (v, v') \in E \} \end{cases}$$

$$\Leftrightarrow \begin{cases} v \in V_{\text{Even}} & \exists (v, v') \in E \quad \mu(v) \geq \delta(\mu(v'), \text{col}(v)) \\ v \in V_{\text{Odd}} & \forall (v, v') \in E \quad \mu(v) \geq \delta(\mu(v'), \text{col}(v)) \end{cases}$$

$$\Leftrightarrow \mu \text{ progress measure}$$

□

### Corollary:

The least progress measure  $\mu_*$  in  $\mathcal{L}$  satisfies

$$\mu_*(v) \neq T \iff v \in W_{\text{Eve}}(G)$$

Proof: follows from the fundamental theorem

$$\mu_*(v) \neq T \Rightarrow v \in W_{\text{Eve}}(G) \text{ follows from (1)}$$

$$v \in W_{\text{Eve}}(G) \Rightarrow \exists \mu \mu(v) \neq T$$

$$\mu_*(v) \leq \mu(v) \Rightarrow \mu_*(v) \neq T \text{ from (2)}$$

□

### Algorithm:

$\mu : v \mapsto \text{leftmost leaf}$

While (Update( $\mu$ )  $\neq \mu$ ):

$\mu \leftarrow \text{Update}(\mu)$

Return  $\{v \in V : \mu(v) \neq T\}$

Number of iterations  $\leq n \cdot |\mathcal{L}|$

Complexity  $O(nm |\mathcal{L}|)$  choose  $\mathcal{L}$  quasipolynomial!