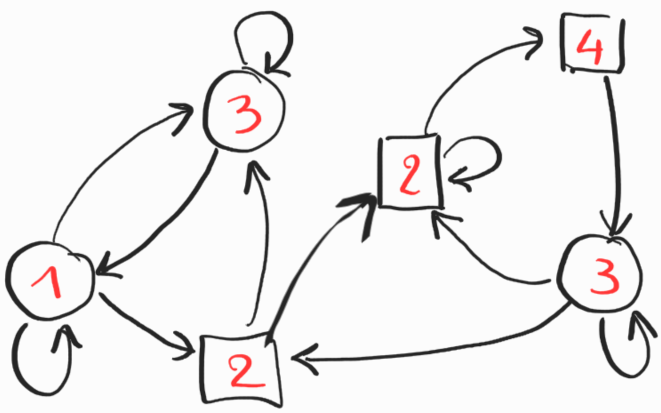
PARITY

Definitions $C = [1, d]$

$\text{PARITY} = \{ \rho \in [1,d]^\omega : \max(\inf(\rho)) \text{ is even} \}$

$\text{CoPARITY} \simeq \text{PARITY} \quad (p \mapsto p+1)$

Example



Theorem Let $\mathcal{G}$ a parity game

$\exists \sigma$ positional strategy winning from $W_{Eve}(\mathcal{G})$

$\exists \tau$ ______________ $W_{Adam}(\mathcal{G}) = V \setminus W_{Eve}(\mathcal{G})$

$W_{Eve}(\mathcal{G})$, σ, and τ, can be computed in $O(m\, n^d)$

Proof induction on d number of priorities

$[1,d]$ two cases: d even and d odd

Since parity is self dual, they are symmetric

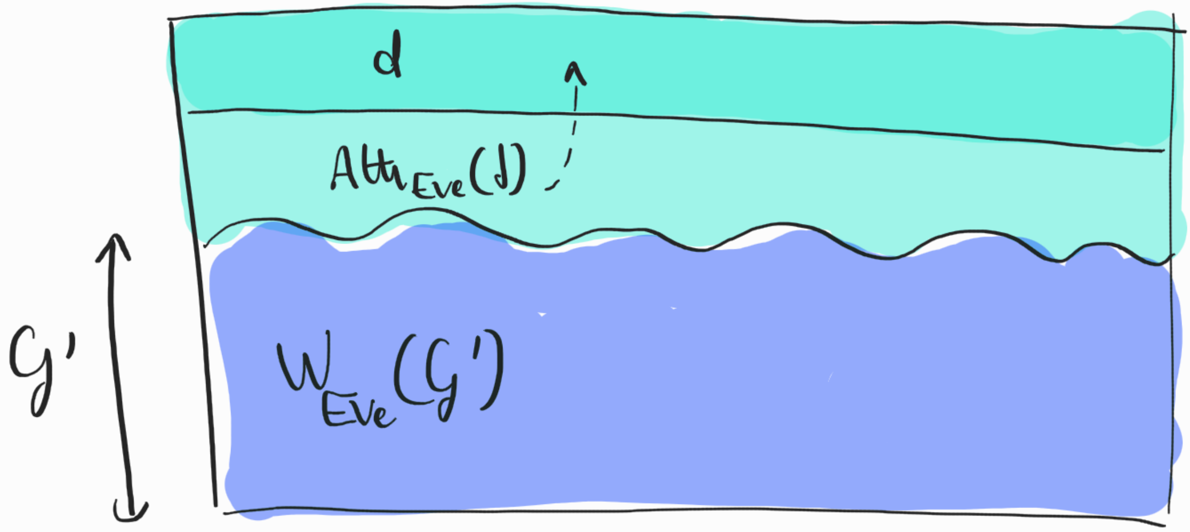
Case d even

$\mathcal{G}$ parity game

$\mathcal{G}' = \mathcal{G}[V \setminus \mathrm{Attr}_{Eve}(d)]$

$\mathcal{G}'$ has priorities $[1, d-1]$ ← induction

1st case: $W_{Adam}(\mathcal{G}') = \emptyset$

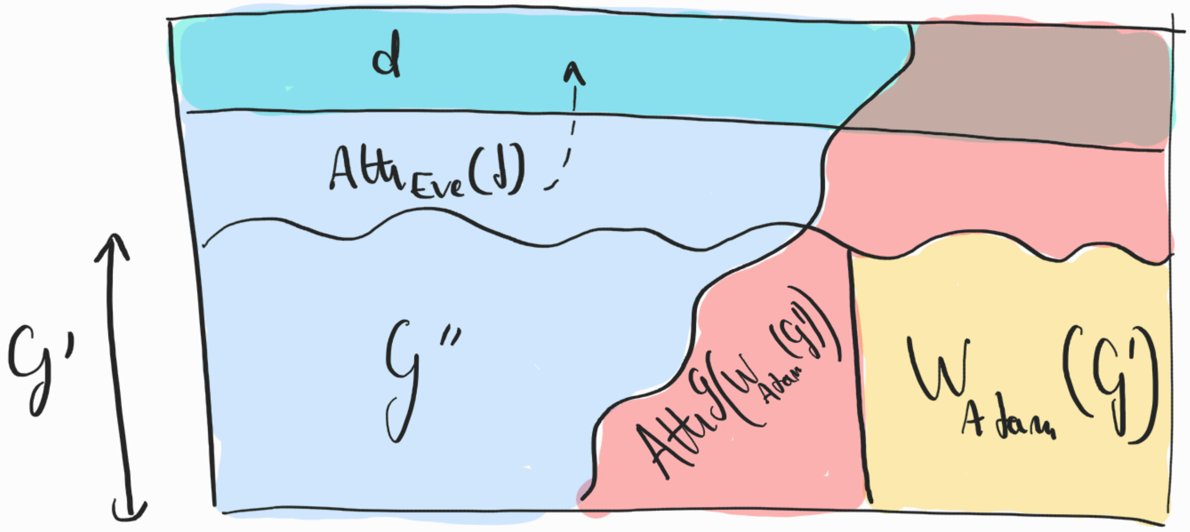


$$\sigma = \begin{cases} \sigma_{Attr} & \text{on } Attr_{Eve}(d) \setminus d \\ \text{anything} & \text{on } d \\ \sigma' & \text{on } W_{Eve}(\mathcal{G}') \end{cases}$$

σ is winning: any play consistent with σ

- either visits d infinitely many times, therefore satisfies parity
- or is eventually consistent with σ', therefore satisfies parity.

2nd case : $W_{Adam}(\mathcal{G}') \neq \emptyset$



$$\mathcal{G}'' = \mathcal{G}\left[V \setminus Attr^{\mathcal{G}}_{Adam}(W_{Adam}(\mathcal{G}'))\right]$$

Claim : $W_E(\mathcal{G}) = W_E(\mathcal{G}'')$

(*) $W_{Eve}(\mathcal{G}'') \subseteq W_{Eve}(\mathcal{G})$

σ winning in $\mathcal{G}''$ is also winning in $\mathcal{G}$:
Adam cannot escape to $Attr^{\mathcal{G}}(W_{Adam}(\mathcal{G}))$

(*) $Attr^{\mathcal{G}}_{Adam}(W_{Adam}(\mathcal{G}')) \subseteq W_{Adam}(\mathcal{G})$

First : $W_{Adam}(\mathcal{G}') \subseteq W_{Adam}(\mathcal{G})$
because Eve cannot escape

Then : closed by attractor : prefix-independent

(*) $W_{Adam}(\mathcal{G}'') \subseteq W_{Adam}(\mathcal{G})$

either wins or escapes to $Attr^{\mathcal{G}}_{Adam}(W_{Adam}(\mathcal{G}'))$

$\square$

Algorithm

If d even (largest priority) : SolveEven($\mathcal{G}$)

If d odd : SolveOdd($\mathcal{G}$)

SolveEven($\mathcal{G}$) :

$X = \mathrm{Attr}^{\mathcal{G}}_{Eve}(d)$

$\mathcal{G}' = \mathcal{G}[V \setminus X]$

$W_{Eve}(\mathcal{G}'), W_{Adam}(\mathcal{G}') \leftarrow \mathrm{SolveOdd}(\mathcal{G}')$

If $W_{Adam}(\mathcal{G}') = \emptyset$:

Return $(V, \emptyset)$

Else :

$Y = \mathrm{Attr}^{\mathcal{G}}_{Adam}(W_{Adam}(\mathcal{G}'))$

$\mathcal{G}'' = \mathcal{G}[V \setminus Y]$

$W_{Eve}(\mathcal{G}''), W_{Adam}(\mathcal{G}'') \leftarrow \mathrm{SolveEven}(\mathcal{G}'')$

Return $(W_{Eve}(\mathcal{G}''), W_{Adam}(\mathcal{G}'') \cup Y)$

Complexity :

$$C(n, m, d) = C(n, m, d-1) + C(n-1, m, d) + O(m)$$

$$O(m\ n^d)$$

Theorem: Solving parity games is in NP $\cap$ coNP

Let G a graph with $col : V \to [1, d]$

- a **cycle is even** if the max priority along the cycle is even
- a **graph satisfies Ω** if all paths in G satisfy Ω

  (Ω = parity or $\Omega = [1,d]^\omega \setminus \text{parity} = \overline{\text{parity}}$)

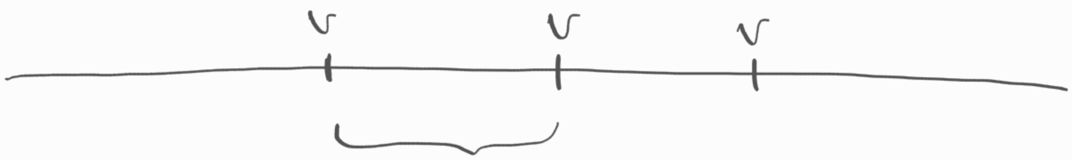
Lemma: G graph over $[1, d]$

G satisfies parity $\iff$ all cycles in G are even

Proof: $\Rightarrow$) clear

$\Leftarrow$) if there exists a path not satisfying parity:

$col(v)$ maximal odd priority appearing infinitely many times



odd cycle

$\square$

Corollary:

(1) Deciding whether a graph satisfies parity can be done in polynomial time

(2) Deciding whether a graph satisfies $\overline{\text{parity}}$ can be done in polynomial time

Theorem: Solving parity games is in NP $\cap$ CoNP

Proof NP: guess σ positional strategy

$G|_\sigma$ graph obtained by restricting to moves of σ

σ winning $\iff$ $G|_\sigma$ satisfies parity

coNP: dual

$\square$