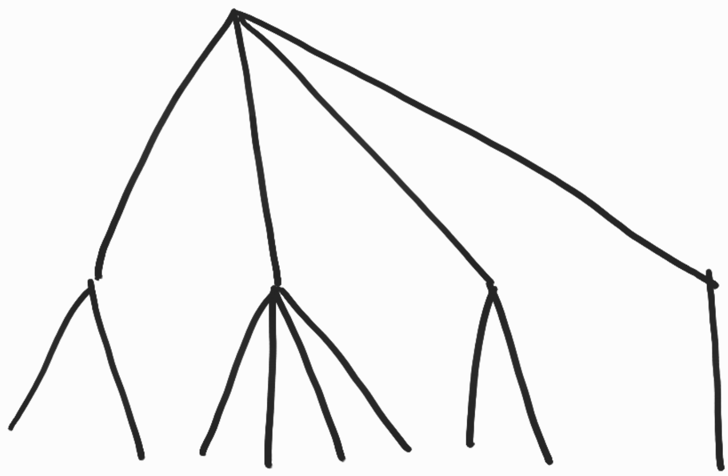
UNIVERSAL TREES

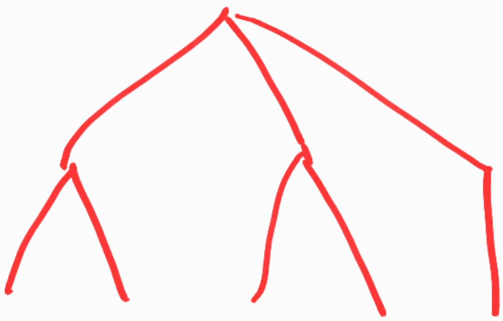
(n, h)-tree :

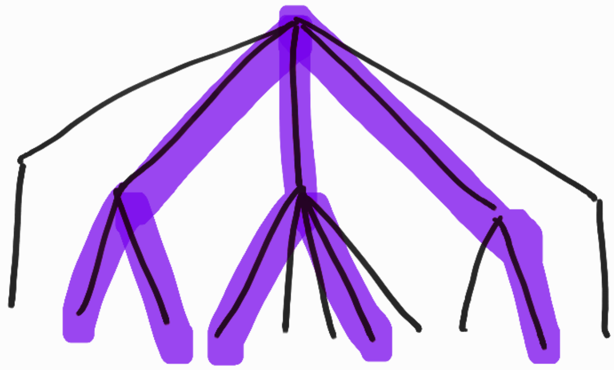
- rooted
- arbitrary degree
- children are ordered
- all leaves have depth h
- at most n leaves

Size : number of leaves



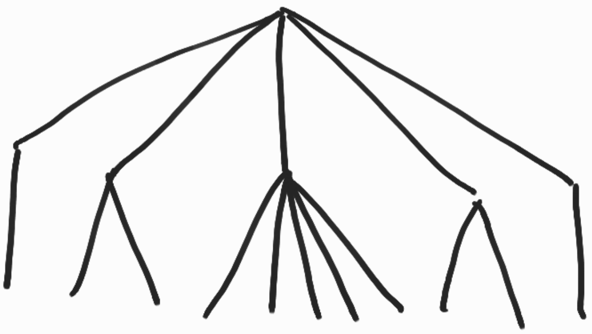
$(9, 2)$-tree





the red tree embeds into the white tree

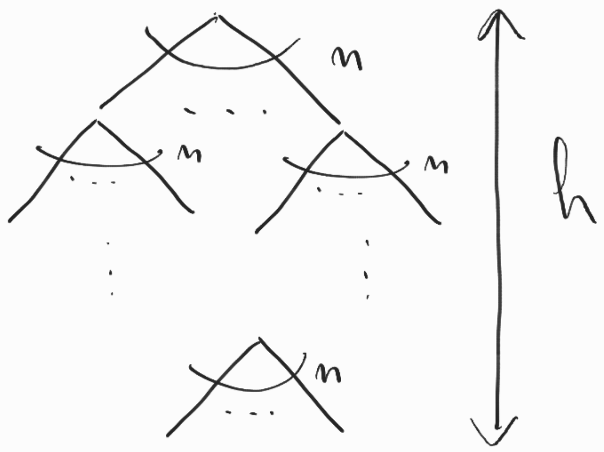
A tree is (n, h)-universal if it embeds all (n, h)-trees



a $(5, 2)$-universal tree

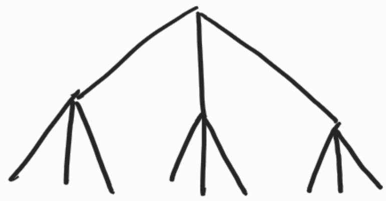
Upper bounds

An exponential construction:



it has n^h leaves

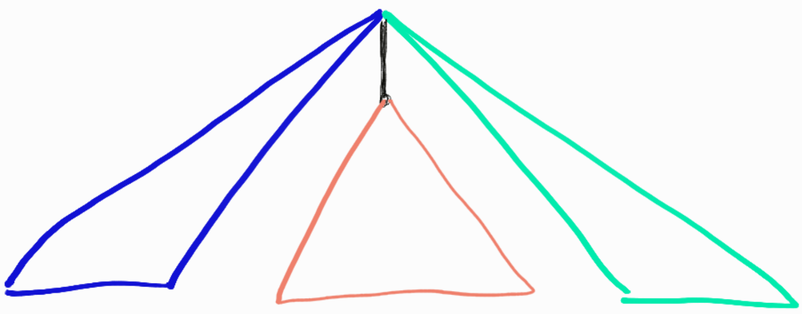
$n = 3 \quad h = 2$



The (n,h)-complete tree

A quasipolynomial construction:

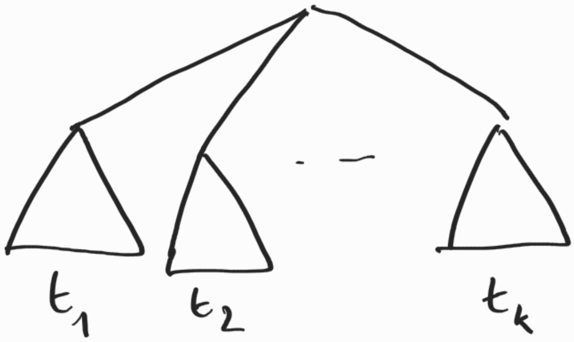
$\mathcal{T}(n,h)$



$\mathcal{T}(n/2,h) \quad \mathcal{T}(n,h-1) \quad \mathcal{T}(n/2,h)$

We show that $\mathcal{T}(n,h)$ is (n,h)-universal

Let t an (n,h)-tree:

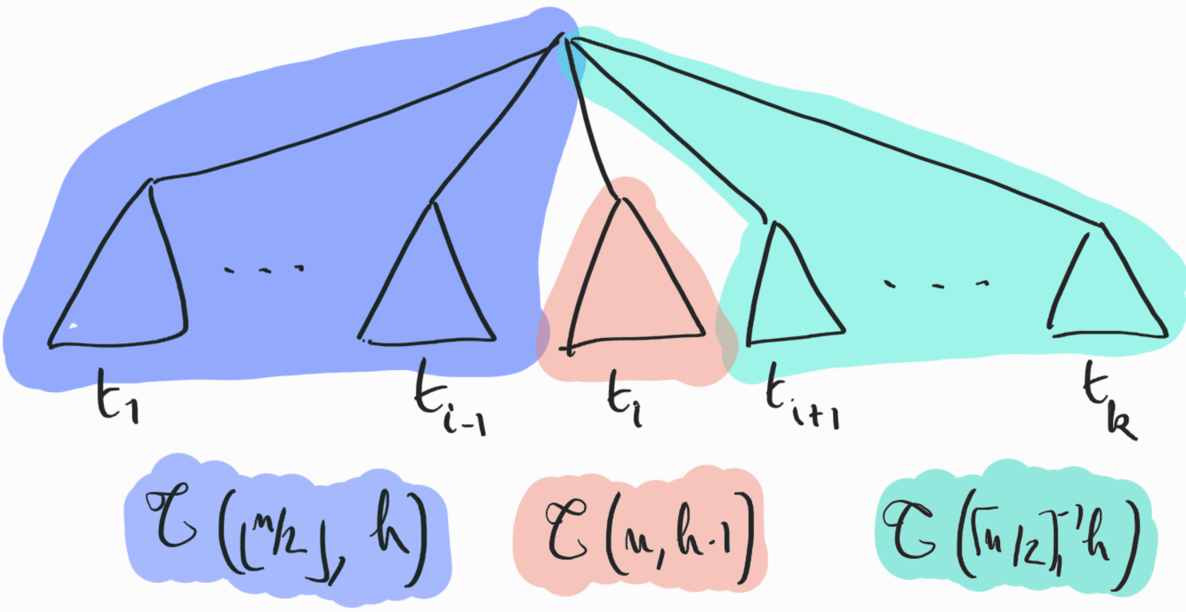


$t_1, \dots, t_k$ have depth $h-1$

$$|t_1| + |t_2| + \dots + |t_k| \leq n$$

$$\Rightarrow \exists i, \quad \begin{aligned} &|t_1| + \dots + |t_{i-1}| \leq n/2 \\ &|t_1| + \dots + |t_i| > n/2 \end{aligned}$$

implying $|t_{i+1}| + \dots + |t_k| \leq n/2$



Size of $\mathcal{C}(n, h)$:

$$|\mathcal{C}(n,h)| = |\mathcal{C}(n/2, h)| + |\mathcal{C}(n, h-1)| + |\mathcal{C}(n/2, h)|$$

$$\mathcal{C}(n,h) \leq n \cdot \binom{h + \log(n)}{h}$$

$$\begin{cases} n^{O(\log h)} & \text{quasipolynomial} \\ n^5 & \text{for } h = O(\log n) \end{cases}$$

Lower bounds

all (n,h)-universal trees have size at least $g(n,h)$
where:

$$g(n,h) = \sum_{\delta=1}^{n} g\left(\lfloor n/\delta \rfloor, h-1\right)$$

$$g(n,h) = n^{\Omega(\log(h))}$$

$$\frac{|\mathcal{T}(n,h)|}{g(n,h)} = O(nh)$$

Conjecture: $\mathcal{T}$ is optimal