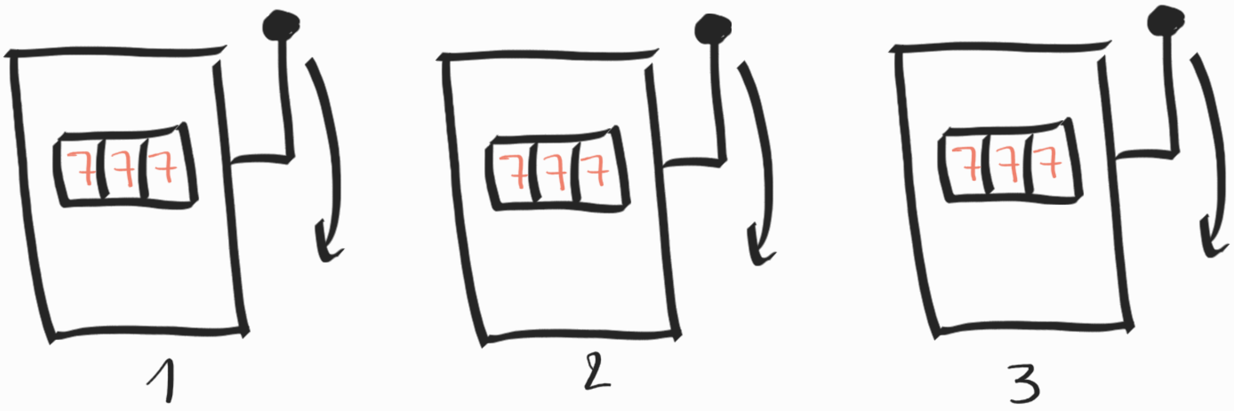
MULTI ARMED BANDITS



arm = machine = action

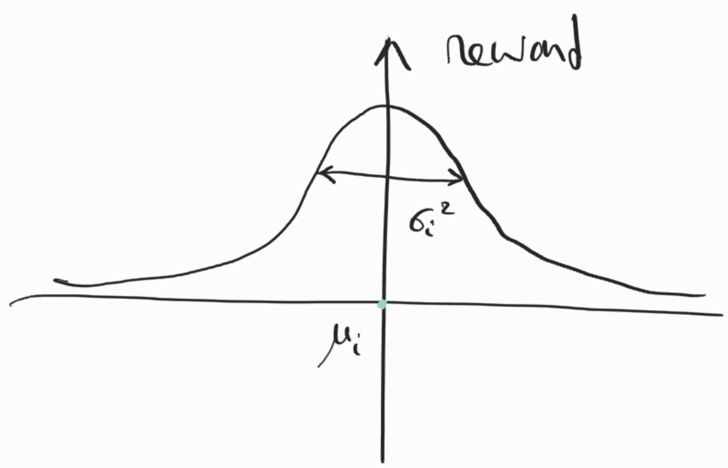
K machines : 1, ..., K

we assume that each machine i has a reward distribution δ_i

Bernoulli distribution $\qquad B(p)$

$$\begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Normal distribution $\qquad \mathcal{N}(\mu, \sigma^2)$



Notation: $\quad \mu = \mathbb{E}[\delta] \qquad \sigma^2 = Var(\delta)$

Scenario:

- we know the number of machines but NOT THE REWARD DISTRIBUTIONS

- at each time step t we pick an arm i and draw a reward $r(t) \in \mathbb{R}$ from δ_i

- Two (similar) goals:
  - → (1) identify the best arm: $\underset{i}{\operatorname{argmax}} \; \mu_i$
  - → (2) maximise the total reward:

$$\sum_{t \geq 1} r(t)$$ ← reward at time t

$K = 2$

$$\delta_1 = \begin{cases} 2 & \text{prob. } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}$$

$$\delta_2 = \begin{cases} 100 & \text{prob. } 1/10 \\ 0 & \text{prob } 9/10 \end{cases}$$

time	machine		
1	1	$\longrightarrow$	$r(1) = 2$
2	2	$\longrightarrow$	$r(2) = 0$
3	2	$\longrightarrow$	$r(3) = 0$
4	2	$\longrightarrow$	$r(4) = 0$

Difficulty: we have to make choices based on incomplete statistics!

Trade off between:

- getting good information on all machines:
  EXPLORATION

- getting good rewards
  EXPLOITATION

after T steps

for each machine $i \in [1, K]$

we have a set of samples

↳ $\hat{\mu}_i(T)$ empirical expectation

Greedy strategy:

EXPLOITATION

play $\underset{i \in [1,K]}{\operatorname{argmax}} \hat{\mu}_i(T)$

ε-Greedy strategy:

ε fixed $\quad \varepsilon = 0.1$

exploration: uniformly at random over all actions — w. probability ε

exploitation: greedy: $\underset{i \in [1,K]}{\operatorname{argmax}} \hat{\mu}_i(T)$ — w. probability $1-\varepsilon$

Notations

$$\underset{i\in[1,K]}{\arg\max}\ \mu_i \stackrel{def}{=} * \in [1,K] \text{ such that } \mu_* = \max_{i\in[1,K]} \mu_i$$

Regret analysis

Regret$(T) = R(T) =$ difference between

"best a posteriori"

and

"what we achieved"

$$R(T) = T \cdot \mu_K - \sum_{t=1}^{T} r(t)$$

⚠ μ_i is the actual expectation

$\hat{\mu}_i(T)$ is the empirical expectation at time T

$$\max \sum_{t \geq 1}^{T} r(t)$$

$$\iff$$

$$\min \quad R(T)$$

We use the regret for comparing different strategies

So far:

Greedy $\quad \arg\max_i \hat{\mu}_i(T)$

ε-Greedy $\quad \begin{cases} \arg\max_i \hat{\mu}_i(T) & \text{with probability } 1-\varepsilon \\ \text{uniform at random} & \text{with probability } \varepsilon \end{cases}$

Issues:
- exploration never stops: at least ε is lost
- exploration does not take existing info into account
- may take a long time to converge

idea: optimistic initialisation

UPPER CONFIDENCE BOUNDS (UCB)

$r(t)$: reward at time t

$r(i,t)$: $\begin{cases} r(t) & \text{if } i \text{ was chosen at time } t \\ 0 & \text{o/w} \end{cases}$

$$R(T) = T \cdot \mu_* - \sum_{t=1}^{T} r(t)$$

$$\hat{\mu}_i(T) = \frac{1}{n(i,T)} \sum_{t=1}^{T} r(i,t)$$

UCB

$$\underset{i}{\operatorname{argmax}} \; \hat{\mu}_i(T) + c(i,T)$$

$$c(i,T) = \sqrt{\frac{\log(T)}{n(i,T)}}$$

intuitions:

- if $n(i,T)$ is small (little information)
  then $c(i,T)$ is large : we need to explore
- if $n(i,T)$ is large
  then $c(i,T)$ is small, except when T grows,
  but exponentially less often

why $c(i,T) = \sqrt{\frac{\log(T)}{n(i,T)}}$?

Chernoff-Hoeffding bounds

let $y_1, \dots, y_n$ iid samples of Y

Law of large numbers: $\frac{1}{n} \sum_{i=1}^{n} y_i \longrightarrow \mathbb{E}[Y]$

Better: this happens fast!

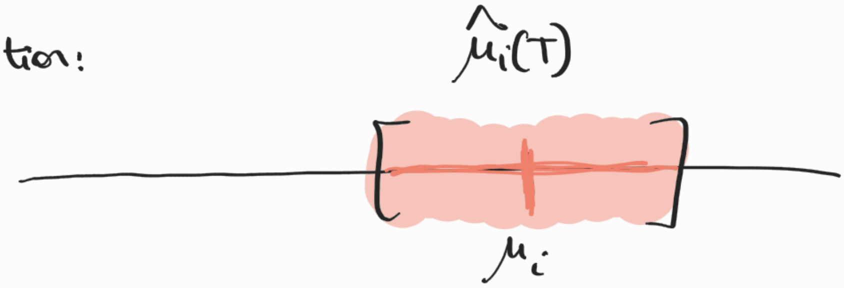
$$\mathbb{P}\left(\left| \frac{1}{n} \sum_{i=1}^{n} y_i - \mathbb{E}[Y] \right| \geq c \right) \leq 2 e^{-2c^2 n}$$

We apply it to our case:

$$\mathbb{P}\left(\left| \hat{\mu}_i(T) - \mu_i \right| \geq c(i,T) \right) \leq 2\, e^{-2 c(i,T)^2 n(i,T)}$$

$c(i,T) = \sqrt{\frac{\log(T)}{n(i,T)}}$ gives $\frac{2}{T^2} \xrightarrow[T \to +\infty]{} 0$

Intuition:



with high probability

UPDATES

$$\hat{\mu}_i(T) = \frac{1}{n(i,T)} \sum_{t=1}^{T} r(i,t)$$

After choosing i :

$$\hat{\mu}_i(T+1) = \frac{1}{\underbrace{n(i,T+1)}_{n(i,T)+1}} \sum_{t=1}^{T+1} r(i,t)$$

small calculations

$$\hat{\mu}_i(T+1) = \hat{\mu}_i(T) + \frac{1}{n(i,T)+1}\left[X(i,T+1) - \hat{\mu}_i(T)\right]$$

$$\text{NEW} = \text{OLD} + \alpha \left[\text{CURRENT} - \text{OLD}\right]$$

we call this step size

Two possible updates :

- empirical mean
- constant step size ($\alpha = 0.1$ for instance)

$$\hat{\mu}_i(T+1) = \hat{\mu}_i(T) + \alpha\left(\hat{\mu}_i(T) - X(i,T+1)\right)$$