Exam MPRI

2021

Note:

- Lecture notes are allowed: using theorems proved during the lectures is expected.
- Although they look similar, the exercises 1 and 2 are independent, and answers from one are not useful or required for the other one.

Definitions for Exercises 1 and 2

This is just a reminder, this is the definitions used in Nathanaël Fijalkow's lectures.

We consider two-player deterministic finite games. An arena $\mathcal{A}$ is given by a set V of vertices with $V = V_{\text{Eve}} \uplus V_{\text{Adam}}$ and a set $E \subseteq V \times V$ of edges. We make the assumption that every vertex has at least one outgoing edge. A winning condition for $\mathcal{A}$ is $W \subseteq V^\omega$. A game $\mathbb{G}$ is a pair $(\mathcal{A}, W)$.

A strategy for Eve is $\sigma : V^* \cdot V_{\text{Eve}} \to E$, and for Adam $\tau : V^+ \cdot V_{\text{Adam}} \to E$. A path is a sequence $v_0 v_1 \dots$ such that for all i we have $(v_i, v_{i+1}) \in E$. It is consistent with σ if for all i, if $v_i \in V_{\text{Eve}}$ then $\sigma(v_0 \dots v_i) = (v_i, v_{i+1})$. The strategy σ is winning from $v \in V$ if all infinite paths π from v consistent with σ satisfy W, meaning $\pi \in W$. In that case we say that v is winning for Eve. Symmetrically we define v being winning for Adam.

We say that $\mathbb{G}$ is determined if for all $v \in V$, either v is winning for Eve or v is winning for Adam. All games we consider are determined (Martin's theorem says that it holds for any Borel objective): we use this result without proving it. We write $W_{\text{Eve}}(\mathbb{G})$ for the set of winning vertices for Eve, and $W_{\text{Adam}}(\mathbb{G})$ for Adam. Then $\mathbb{G}$ is determined if $W_{\text{Eve}}(\mathbb{G}) \cup W_{\text{Adam}}(\mathbb{G}) = V$.

A positional strategy for Eve is $\sigma : V_{\text{Eve}} \to E$, and for Adam $\tau : V_{\text{Adam}} \to E$. We say that $\mathbb{G}$ is positionally determined for Eve if for all $v \in W_{\text{Eve}}(\mathbb{G})$, there exists a positional winning strategy from v. Similarly for Adam.

An objective is $\Omega \subseteq C^\omega$ with C a set of colours. The objective Ω and a colouring function $\text{col} : V \to C$ (we colour vertices) induce a condition $\Omega[\text{col}] \subseteq V^\omega$:

$$\Omega[\text{col}] = \{v_0 v_1 \cdots : \text{col}(v_0)\text{col}(v_1) \cdots \in \Omega\}.$$

We say that $\mathbb{G} = (\mathcal{A}, \Omega[\text{col}])$ has objective Ω, and that:

- Ω is prefix independent if for all $w \in C^*, w' \in C^\omega$ we have $w' \in \Omega \iff ww' \in \Omega$.
- Ω is positionally determined for Eve if all games with objectives Ω are positionally determined.
- Ω is positionally determined if it holds for both Eve and Adam.

In evaluating algorithms the important parameters from the graph are n the number of vertices and m the number of edges.

Exercise 1

Let $C = [1, d]$ for $d \in \mathbb{N}$. We define the weak parity objective:

$$\mathsf{WeakParity} = \{\rho \in [1, d]^\omega : \max(\rho) \text{ is even}\}.$$

Question 1: Prove or disprove: $\mathsf{WeakParity}$ is prefix independent.

Let $\mathbb{G}$ be a game with objective $\mathsf{WeakParity}$: $\mathbb{G} = (\mathcal{A}, \mathsf{WeakParity}[\mathrm{col}])$.
Let $V_d = \{v \in V : \mathrm{col}(v) = d\}$. Let us assume that d is even.

Question 2: Show that if $\mathrm{Attr}_{\mathrm{Eve}}(V_d) = V$, then $W_{\mathrm{Eve}}(\mathbb{G}) = V$.

For $F \subseteq V$ we define the reachability condition $\mathsf{Reach}(F) = \{v_0 v_1 \cdots : \exists i \in \mathbb{N}, v_i \in F\}$. We write $\mathrm{Attr}_{\mathrm{Eve}}(F)$ for $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{Reach}(F))$.

Question 3: Define the game induced from $\mathbb{G}$ by $V \backslash \mathrm{Attr}_{\mathrm{Eve}}(F)$ and show that in this induced game, every vertex has at least one outgoing edge (so it is well defined).

Question 4: Let $\mathbb{G}'$ the game induced from $\mathbb{G}$ by $V \backslash \mathrm{Attr}_{\mathrm{Eve}}(d)$. Show that $W_{\mathrm{Eve}}(\mathbb{G}) = \mathrm{Attr}_{\mathrm{Eve}}(V_d) \cup W_{\mathrm{Eve}}(\mathbb{G}')$.

Question 5: Construct an algorithm for solving weak parity games (meaning computing the set of winning vertices for Eve) and evaluate its complexity.

Question 6: Prove or disprove (for both players): $\mathsf{WeakParity}$ is positionally determined.

Exercise 2

Let $C = [1, k]$ for $k \in \mathbb{N}$. We define the generalized CoBüchi objective:

$$\mathsf{GenCoBuchi} = \{\rho \in [1, k]^\omega : \exists i \in [1, k], i \notin \inf(\rho)\},$$

where $\inf(\rho)$ is the set of colours appearing infinitely many times in ρ.

Question 1: Prove or disprove: $\mathsf{GenCoBuchi}$ is prefix independent.

Question 2: Prove or disprove (for both players): $\mathsf{GenCoBuchi}$ is positionally determined.

Let $\mathbb{G}$ be a game with objective $\mathsf{GenCoBuchi}$: $\mathbb{G} = (\mathcal{A}, \mathsf{GenCoBuchi}[\mathrm{col}])$. We write $V_i = \{v \in V : \mathrm{col}(v) = i\}$, and for $F \subseteq V$ we write $\mathsf{CoBuchi}(F) = \{\pi \in V^\omega : \inf(\rho) \cap F = \emptyset\}$.

Question 3: Prove or disprove: $W_{\mathrm{Eve}}(\mathbb{G}) = \bigcup_{i \in [1,k]} W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i))$.

Question 4: Show that if for all $i \in [1, k]$ we have $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i)) = \emptyset$, then $W_{\mathrm{Eve}}(\mathbb{G}) = \emptyset$.

Question 5: Assume that for some $i \in [1, k]$ we have $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i)) \neq \emptyset$, show that the game $\mathbb{G}'$ induced from $\mathbb{G}$ by $V \backslash W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i))$ is well defined, and that $W_{\mathrm{Eve}}(\mathbb{G}) = W_{\mathrm{Eve}}(\mathbb{G}') \cup W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i))$.

Question 6: Construct an algorithm for solving generalized CoBüchi games (meaning computing the set of winning vertices for Eve) and evaluate its complexity.