# Exam MPRI

2023

Lecture notes are allowed: using theorems proved during the lectures is expected.

## Definitions for Exercise 1

This is just a reminder, this is the definitions used in Nathanaël Fijalkow's lectures.

We consider two-player deterministic finite games. An arena $\mathcal{A}$ is given by a set $V$ of vertices with $V = V_{\text{Eve}} \uplus V_{\text{Adam}}$ and a set $E \subseteq V \times V$ of edges. We make the assumption that every vertex has at least one outgoing edge. A winning condition for $\mathcal{A}$ is $W \subseteq V^\omega$. A game $\mathbb{G}$ is a pair $(\mathcal{A}, W)$.

A strategy for Eve is $\sigma : V^* \cdot V_{\text{Eve}} \to E$, and for Adam $\tau : V^+ \cdot V_{\text{Adam}} \to E$. A path is a sequence $v_0 v_1 \dots$ such that for all $i$ we have $(v_i, v_{i+1}) \in E$. It is consistent with $\sigma$ if for all $i$, if $v_i \in V_{\text{Eve}}$ then $\sigma(v_0 \dots v_i) = (v_i, v_{i+1})$. The strategy $\sigma$ is winning from $v \in V$ if all infinite paths $\pi$ from $v$ consistent with $\sigma$ satisfy $W$, meaning $\pi \in W$. In that case we say that $v$ is winning for Eve. Symmetrically we define $v$ being winning for Adam.

We say that $\mathbb{G}$ is determined if for all $v \in V$, either $v$ is winning for Eve or $v$ is winning for Adam. All games we consider are determined (Martin's theorem says that it holds for any Borel objective): we use this result without proving it. We write $W_{\text{Eve}}(\mathbb{G})$ for the set of winning vertices for Eve, and $W_{\text{Adam}}(\mathbb{G})$ for Adam. Then $\mathbb{G}$ is determined if $W_{\text{Eve}}(\mathbb{G}) \cup W_{\text{Adam}}(\mathbb{G}) = V$.

A positional strategy for Eve is $\sigma : V_{\text{Eve}} \to E$, and for Adam $\tau : V_{\text{Adam}} \to E$. We say that $\mathbb{G}$ is positionally determined for Eve if for all $v \in W_{\text{Eve}}(\mathbb{G})$, there exists a positional winning strategy from $v$. Similarly for Adam.

An objective is $\Omega \subseteq C^\omega$ with $C$ a set of colours. The objective $\Omega$ and a colouring function $\text{col} : V \to C$ (we colour vertices) induce a condition $\Omega[\text{col}] \subseteq V^\omega$:

$$\Omega[\text{col}] = \{v_0 v_1 \dots : \text{col}(v_0)\text{col}(v_1) \dots \in \Omega\}.$$

We say that $\mathbb{G} = (\mathcal{A}, \Omega[\text{col}])$ has objective $\Omega$, and that:

- $\Omega$ is prefix independent if for all $w \in C^*, w' \in C^\omega$ we have $w' \in \Omega \iff ww' \in \Omega$.
- $\Omega$ is positionally determined for Eve if all games with objectives $\Omega$ are positionally determined.
- $\Omega$ is positionally determined if it holds for both Eve and Adam.

In evaluating algorithms the important parameters from the graph are $n$ the number of vertices and $m$ the number of edges.

## Exercise 1: Finitary parity games

Let $C = [1, d]$ for $d \in \mathbb{N}$. Let us fix an arena $\mathcal{A}$ and $\mathrm{col} : V \to C$.

We define the finitary parity objective:

$$\mathsf{FinitaryParity} = \{\rho \in [1, d]^\omega : \exists N, B \in \mathbb{N}, \forall i \geqslant N, \exists j \in [i, i + B], \rho_i \leqslant \rho_j \text{ and } \rho_j \text{ even}\}.$$

Two examples:

- $3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdots \in \mathsf{FinitaryParity}$
- $4 \cdot (3 \cdot 2 \cdot 4) \cdot (3 \cdot \underbrace{2 \cdot 2}_{2 \text{ times}} \cdot 4) \ldots (3 \cdot \underbrace{2 \cdot 2 \ldots 2}_{n \text{ times}} \cdot 4) \cdots \notin \mathsf{FinitaryParity}$

**Question 1:** Prove or disprove:

- $\mathsf{FinitaryParity}$ is prefix independent,
- $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{Parity}[\mathrm{col}]) \subseteq W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{FinitaryParity}[\mathrm{col}])$,
- $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{FinitaryParity}[\mathrm{col}]) \subseteq W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{Parity}[\mathrm{col}])$,
- $\mathsf{FinitaryParity}$ is positionally determined for Adam.

We define the bounded parity objective:

$$\mathsf{BoundedParity} = \{\rho \in [1, d]^\omega : \exists B \in \mathbb{N}, \forall i \in \mathbb{N}, \exists j \in [i, i + B], \rho_i \leqslant \rho_j \text{ and } \rho_j \text{ even}\}.$$

**Question 2:** Show that

- $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{BoundedParity}[\mathrm{col}]) \subseteq W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{FinitaryParity}[\mathrm{col}])$,
- if $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{BoundedParity}[\mathrm{col}]) = \emptyset$, then $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{FinitaryParity}[\mathrm{col}]) = \emptyset$.

**Question 3:** Assuming an algorithm for solving bounded parity games, construct an algorithm for solving finitary parity games. What is the complexity of the algorithm (as a function of the complexity of the algorithm for bounded parity games)?

We define the weak parity objective:

$$\mathsf{WeakParity} = \{\rho \in [1, d]^\omega : \max(\rho) \text{ is even}\}.$$

**Question 4:** Show that

- $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{BoundedParity}[\mathrm{col}]) \subseteq W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{WeakParity}[\mathrm{col}])$,
- if $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{WeakParity}[\mathrm{col}]) = V$, then $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{BoundedParity}[\mathrm{col}]) = V$.

**Question 5:** Assuming an algorithm for solving weak parity games, construct an algorithm for solving bounded parity games. What is the complexity of the algorithm (as a function of the complexity of the algorithm for weak parity games)?

Let us now construct an algorithm for solving weak parity games. Let $V_d = \{v \in V : \mathrm{col}(v) = d\}$. We assume that $d$ is even.

**Question 6:** Show that if $\mathrm{Attr}_{\mathrm{Eve}}(V_d) = V$, then $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{WeakParity}[\mathrm{col}]) = V$.

**Question 7:** Let $\mathbb{G}'$ the weak parity game induced from $\mathbb{G}$ by $V \setminus \text{Attr}_{\text{Eve}}(d)$. Show that $W_{\text{Eve}}(\mathbb{G}) = \text{Attr}_{\text{Eve}}(V_d) \cup W_{\text{Eve}}(\mathbb{G}')$.

**Question 8:** Construct an algorithm for solving weak parity games (meaning computing the set of winning vertices for Eve) and evaluate its complexity.

**Question 9:** What is the complexity of solving finitary parity games?

**Question 10:** Prove or disprove:

- $\mathsf{WeakParity}$ is positionally determined for Eve,
- $\mathsf{BoundedParity}$ is positionally determined for Eve,
- $\mathsf{FinitaryParity}$ is positionally determined for Eve.