Exam MPRI

2021

Exercise 1

Let $C = [1, d]$ for $d \in \mathbb{N}$. We define the weak parity objective:

$$\mathsf{WeakParity} = \{\rho \in [1, d]^\omega : \max(\rho) \text{ is even}\}.$$

Question 1: Prove or disprove: $\mathsf{WeakParity}$ is prefix independent.

Solution: Not prefix independent: $2^\omega \in \mathsf{WeakParity}$ but $32^\omega \notin \mathsf{WeakParity}$.

Let $\mathbb{G}$ be a game with objective $\mathsf{WeakParity}$: $\mathbb{G} = (\mathcal{A}, \mathsf{WeakParity}[\text{col}])$.
Let $V_d = \{v \in V : \text{col}(v) = d\}$. Let us assume that d is even.

Question 2: Show that if $\text{Attr}_{\text{Eve}}(V_d) = V$, then $W_{\text{Eve}}(\mathbb{G}) = V$.

Solution: Eve plays the attractor strategy from $\text{Attr}_{\text{Eve}}(V_d)\backslash V_d$, and anything from V_d. This way she ensures to see d from anywhere, maximal and even, hence $\mathsf{WeakParity}$ is satisfied.

For $F \subseteq V$ we define the reachability condition $\mathsf{Reach}(F) = \{v_0 v_1 \cdots : \exists i \in \mathbb{N}, v_i \in F\}$. We write $\text{Attr}_{\text{Eve}}(F)$ for $W_{\text{Eve}}(\mathcal{A}, \mathsf{Reach}(F))$.

Question 3: Define the game induced from $\mathbb{G}$ by $V\backslash\text{Attr}_{\text{Eve}}(F)$ and show that in this induced game, every vertex has at least one outgoing edge (so it is well defined).

Solution: The definition is clear. The point is that for any vertex controlled by Eve in $V\backslash\text{Attr}_{\text{Eve}}(F)$, there are no outgoing edges to $\text{Attr}_{\text{Eve}}(F)$, and for any vertex controlled by Adam in $V\backslash\text{Attr}_{\text{Eve}}(F)$ there is at least one outgoing edge to $V\backslash\text{Attr}_{\text{Eve}}(F)$.

Question 4: Let $\mathbb{G}'$ the game induced from $\mathbb{G}$ by $V\backslash\text{Attr}_{\text{Eve}}(d)$. Show that $W_{\text{Eve}}(\mathbb{G}) = \text{Attr}_{\text{Eve}}(V_d) \cup W_{\text{Eve}}(\mathbb{G}')$.

Solution: We prove both inclusions. First, $\text{Attr}_{\text{Eve}}(V_d) \cup W_{\text{Eve}}(\mathbb{G}') \subseteq W_{\text{Eve}}(\mathbb{G})$: from $\text{Attr}_{\text{Eve}}(V_d)$ Eve wins by attracting to d, the largest even priority, and from $W_{\text{Eve}}(\mathbb{G}')$ Eve wins in the game $\mathbb{G}$ because the play either stays in $\mathbb{G}'$ and is winning, or enters $\text{Attr}_{\text{Eve}}(V_d)$ from where Eve wins. Conversely, we show that $W_{\text{Eve}}(\mathbb{G}) \subseteq \text{Attr}_{\text{Eve}}(V_d) \cup W_{\text{Eve}}(\mathbb{G}')$ by considering the complement: we show that $W_{\text{Adam}}(\mathbb{G}) \subseteq W_{\text{Adam}}(\mathbb{G}')$. Indeed, Adam cannot win from $\text{Attr}_{\text{Eve}}(V_d)$, so he can only win from $\mathbb{G}'$.

Question 5: Construct an algorithm for solving weak parity games (meaning computing the set of winning vertices for Eve) and evaluate its complexity.

Solution: The algorithm follows from the two previous questions, plus dual cases. Compute the attractor to the maximal priority d appearing in the game. If the attractor is the whole game, then the corresponding player (Eve if the d is even, Adam if d is odd) wins everywhere. Otherwise, remove the attractor from the game, declare this subset of vertices winning for the corresponding player, and continue recursively.

Naive complexity analysis: n iterations, each costs $O(m)$, so $O(nm)$.

Question 6: Prove or disprove (for both players): $\mathsf{WeakParity}$ is positionally determined.

Solution: First, the complement of $\mathsf{WeakParity}$ is another $\mathsf{WeakParity}$, by shifting the priorities by one, so if Eve is positionnally determined, so is Adam. The algorithm above and its proof reveals that indeed both players have positional winning strategies.

Exercise 2

Let $C = [1, k]$ for $k \in \mathbb{N}$. We define the generalized CoBüchi objective:

$$\mathsf{GenCoBuchi} = \{\rho \in [1, k]^\omega : \exists i \in [1, k], i \notin \inf(\rho)\},$$

where $\inf(\rho)$ is the set of colours appearing infinitely many times in ρ.

Question 1: Prove or disprove: $\mathsf{GenCoBuchi}$ is prefix independent.

Solution: Prefix independent, it only considers colours seen infinitely many times.

Question 2: Prove or disprove (for both players): $\mathsf{GenCoBuchi}$ is positionally determined.

Solution: A direct application of the submixing theorem yields that $\mathsf{GenCoBuchi}$ is positionally determined. However, its complement is not: consider the game where Adam can choose between seing 1 and seeing 2. In order to win to needs to see them both infinitely many times, so he needs two memory states.

Let $\mathbb{G}$ be a game with objective $\mathsf{GenCoBuchi}$: $\mathbb{G} = (\mathcal{A}, \mathsf{GenCoBuchi}[\mathsf{col}])$. We write $V_i = \{v \in V : \mathsf{col}(v) = i\}$, and for $F \subseteq V$ we write $\mathsf{CoBuchi}(F) = \{\pi \in V^\omega : \inf(\rho) \cap F = \emptyset\}$.

Question 3: Prove or disprove: $W_{\mathrm{Eve}}(\mathbb{G}) = \bigcup_{i \in [1,k]} W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i))$.

Solution: Unfortunately not true. Consider the game where Adam can choose between two vertices: in the first one Eve sees only colour 1, and in the second only colour 2. Eve indeed wins from everywhere, but the initial vertex does not belong to any $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i))$. Intuitively, she wins, but does not know with which colour she will win.

Question 4: Show that if for all $i \in [1, k]$ we have $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i)) = \emptyset$, then $W_{\mathrm{Eve}}(\mathbb{G}) = \emptyset$.

Solution: Assume that for all $i \in [1, k]$ we have $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i)) = \emptyset$, so Adam has a strategy τ_i to see colour i infinitely many times from anywhere. We construct a strategy τ as follows: it plays τ_1 until it reaches colour 1, then τ_2, and so on. This strategy ensures to see all colours infinitely many times from anywhere, so it wins everywhere, implying that $W_{\mathrm{Eve}}(\mathbb{G}) = \emptyset$.

Question 5: Assume that for some $i \in [1, k]$ we have $W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i)) \neq \emptyset$, show that the game $\mathbb{G}'$ induced from $\mathbb{G}$ by $V \setminus W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i))$ is well defined, and that $W_{\mathrm{Eve}}(\mathbb{G}) = W_{\mathrm{Eve}}(\mathbb{G}') \cup W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{CoBuchi}(V_i))$.

Solution: The game is well defined: every vertex has an outgoing edge. We prove both inclusions, similar as in exercise 1.

Question 6: Construct an algorithm for solving generalized CoBüchi games (meaning computing the set of winning vertices for Eve) and evaluate its complexity.

Solution: The algorithm follows from the two previous questions. Compute for each i the winning region for the objective $\mathsf{CoBuchi}(V_i)$. If for some i, the winning region is non empty, declare this subset of vertices winning for Eve, remove it from the game, and continue recursively. Otherwise, Adam wins everywhere.

Naive complexity analysis: n iterations, each costs $O(knm)$, so $O(kn^2m)$.