Exam MPRI

24 November 2022

Note:

- Lecture notes are allowed: using theorems proved during the lectures is expected.
- The exercises 1 and 2 are independent.

Definitions for Exercises 1 and 2

This is just a reminder, this is the definitions used in Nathanaël Fijalkow's lectures.

We consider two-player deterministic finite games. An arena $\mathcal{A}$ is given by a set V of vertices with $V = V_{\text{Eve}} \uplus V_{\text{Adam}}$ and a set $E \subseteq V \times V$ of edges. We make the assumption that every vertex has at least one outgoing edge. A winning condition for $\mathcal{A}$ is $W \subseteq V^\omega$. A game $\mathbb{G}$ is a pair $(\mathcal{A}, W)$.

A strategy for Eve is $\sigma : V^* \cdot V_{\text{Eve}} \to E$, and for Adam $\tau : V^+ \cdot V_{\text{Adam}} \to E$. A path is a sequence $v_0 v_1 \ldots$ such that for all i we have $(v_i, v_{i+1}) \in E$. It is consistent with σ if for all i, if $v_i \in V_{\text{Eve}}$ then $\sigma(v_0 \ldots v_i) = (v_i, v_{i+1})$. The strategy σ is winning from $v \in V$ if all infinite paths π from v consistent with σ satisfy W, meaning $\pi \in W$. In that case we say that v is winning for Eve. Symmetrically we define v being winning for Adam.

We say that $\mathbb{G}$ is determined if for all $v \in V$, either v is winning for Eve or v is winning for Adam. All games we consider are determined (Martin's theorem says that it holds for any Borel objective): we use this result without proving it. We write $W_{\text{Eve}}(\mathbb{G})$ for the set of winning vertices for Eve, and $W_{\text{Adam}}(\mathbb{G})$ for Adam. Then $\mathbb{G}$ is determined if $W_{\text{Eve}}(\mathbb{G}) \cup W_{\text{Adam}}(\mathbb{G}) = V$.

A positional strategy for Eve is $\sigma : V_{\text{Eve}} \to E$, and for Adam $\tau : V_{\text{Adam}} \to E$. We say that $\mathbb{G}$ is positionally determined for Eve if for all $v \in W_{\text{Eve}}(\mathbb{G})$, there exists a positional winning strategy from v. Similarly for Adam.

An objective is $\Omega \subseteq C^\omega$ with C a set of colours. The objective Ω and a colouring function $\text{col} : V \to C$ (we colour vertices) induce a condition $\Omega[\text{col}] \subseteq V^\omega$:

$$\Omega[\text{col}] = \{v_0 v_1 \cdots : \text{col}(v_0)\text{col}(v_1) \cdots \in \Omega\}.$$

We say that $\mathbb{G} = (\mathcal{A}, \Omega[\text{col}])$ has objective Ω, and that:

- Ω is prefix independent if for all $w \in C^*, w' \in C^\omega$ we have $w' \in \Omega \iff ww' \in \Omega$.
- Ω is positionally determined for Eve if all games with objectives Ω are positionally determined.
- Ω is positionally determined if it holds for both Eve and Adam.

In evaluating algorithms the important parameters from the graph are n the number of vertices and m the number of edges.

Exercise 1

Let $C = [1, 2d]$ for $d \in \mathbb{N}$. We define the Rabin objective:

$$\mathsf{Rabin} = \left\{\rho \in [1, 2d]^\omega : \exists i \in [1, d], 2i \in \inf(\rho) \wedge 2i + 1 \notin \inf(\rho)\right\},$$

where $\inf(\rho)$ is the set of colours appearing infinitely many times in ρ.

Question 1: Prove or disprove: Rabin is prefix independent.

Solution 1: It is prefix independent because it only looks at which colours appear infinitely many times.

Let us fix a game $\mathbb{G} = (\mathcal{A}, \Omega[\mathrm{col}])$. For $F \subseteq V$ the reachability condition is

$$\mathsf{Reach}(F) = \{v_0 v_1 \cdots : \exists i \in \mathbb{N}, v_i \in F\}.$$

We write

$$\mathrm{Attr}^{\mathbb{G}}_{\mathrm{Eve}}(F) = W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{Reach}(F))$$

and similarly $\mathrm{Attr}^{\mathbb{G}}_{\mathrm{Adam}}(F) = W_{\mathrm{Adam}}(\mathcal{A}, \mathsf{Reach}(F)) = V \setminus W_{\mathrm{Eve}}(\mathcal{A}, \mathsf{Reach}(F))$. The index $\mathbb{G}$ is used to specify that the attractor computation is performed in $\mathbb{G}$ (note that it only depends on $\mathcal{A}$). Since there will be many different games in the rest of this exercise, it is important to specify this information. Given a game $\mathbb{G}$ and a subset F, we define the subgame of $\mathbb{G}$ where we remove $\mathrm{Attr}^{\mathbb{G}}_{\mathrm{Eve}}(F)$ in the expected way, and the same for $\mathrm{Attr}^{\mathbb{G}}_{\mathrm{Adam}}(F)$.

We now construct an algorithm for solving Rabin games, meaning computing the set of winning vertices for Eve and Adam. Let $\mathbb{G}$ be a Rabin game, we define $R_i = \{v \in V : \mathrm{col}(v) = 2i\}$ and $G_i = \{v \in V : \mathrm{col}(v) = 2i + 1\}$.

Let us define $\mathbb{G}_i$ the subgame of $\mathbb{G}$ where we remove $\mathrm{Attr}^{\mathbb{G}}_{\mathrm{Adam}}(G_i)$, and $\mathbb{G}'_i$ the subgame of $\mathbb{G}_i$ where we remove $\mathrm{Attr}^{\mathbb{G}_i}_{\mathrm{Eve}}(R_i)$.

Question 2: Show that if $W_{\mathrm{Adam}}(\mathbb{G}'_i) = \emptyset$, then $W_{\mathrm{Adam}}(\mathbb{G}_i) = \emptyset$.

Solution 2: Let σ_i be a winning strategy in $\mathbb{G}'_i$ and σ_A an attractor strategy ensuring to reach a vertex in R_i from any vertex in $\mathrm{Attr}^{\mathbb{G}_i}_{\mathrm{Eve}}(R_i) \setminus R_i$. We construct a strategy in $\mathbb{G}_i$ as follows: as long as the play remains in $\mathbb{G}'_i$ Eve plays using σ_i, and if it leaves $\mathbb{G}'_i$ then Eve plays σ_A until reaching R_i. This strategy is winning: a play consistent with it either sees infinitely many times R_i or is eventually consistent with σ_i, which implies in both cases that it satisfies Rabin.

Define $\mathbb{G}''_i$ the subgame of $\mathbb{G}_i$ where we remove $\mathrm{Attr}^{\mathbb{G}_i}_{\mathrm{Adam}}(W_{\mathrm{Adam}}(\mathbb{G}'_i))$,

Question 3: Show that if $W_{\mathrm{Adam}}(\mathbb{G}'_i) \neq \emptyset$, then $W_{\mathrm{Eve}}(\mathbb{G}_i) = W_{\mathrm{Eve}}(\mathbb{G}''_i)$.

Solution 3: We prove both inclusions.

We show $W_{\mathrm{Eve}}(\mathbb{G}_i) \subseteq W_{\mathrm{Eve}}(\mathbb{G}''_i)$. We actually show the contrapositive: $W_{\mathrm{Adam}}(\mathbb{G}''_i) \subset W_{\mathrm{Adam}}(\mathbb{G}_i)$. We construct a strategy τ for Adam in $\mathbb{G}'_i$ as follows:

- On $\mathrm{Attr}^{\mathbb{G}_i}_{\mathrm{Adam}}(W_{\mathrm{Adam}}(\mathbb{G}'_i)) \setminus W_{\mathrm{Adam}}(\mathbb{G}'_i)$ we play the attractor strategy for Adam to reach $W_{\mathrm{Adam}}(\mathbb{G}'_i)$;
- On $W_{\mathrm{Adam}}(\mathbb{G}'_i)$ we play the winning strategy of Adam in $\mathbb{G}'_i$;
- On $W_{\mathrm{Adam}}(\mathbb{G}''_i)$ we play the winning strategy of Adam in $\mathbb{G}''_i$.

This strategy is winning: a play consistent with it either eventually enters $\mathrm{Attr}^{\mathbb{G}_i}_{\mathrm{Adam}}(W_{\mathrm{Adam}}(\mathbb{G}'_i))$, and then it is winning because it later enters $W_{\mathrm{Adam}}(\mathbb{G}'_i)$ and remains there and is consistent with a winning strategy, or it remains forever in $W_{\mathrm{Adam}}(\mathbb{G}''_i)$, which implies in both cases that it does not satisfy Rabin.

We now show $W_{\text{Eve}}(\mathbb{G}_i) \supseteq W_{\text{Eve}}(\mathbb{G}''_i)$. Let σ be a winning strategy in $W_{\text{Eve}}(\mathbb{G}''_i)$. It induces a winning strategy in $\mathbb{G}_i$: indeed if Eve uses σ in $\mathbb{G}_i$, then the play remains in $W_{\text{Eve}}(\mathbb{G}''_i)$ so it is winning.

Question 4: Show that if for all $i \in [1,d]$, $W_{\text{Eve}}(\mathbb{G}_i) = \emptyset$, then $W_{\text{Adam}}(\mathbb{G}) = V$.

Solution 4: We construct a winning strategy for Adam using a memory: the set of memory states is $[1,d]$, and in the state i:

- On $\text{Attr}^{\mathbb{G}}_{\text{Adam}}(G_i)$, Adam plays the attractor strategy to reach a vertex in G_i. When reaching G_i we update the memory state to $i+1$ (or 1 if $i = d$).
- On $V \setminus \text{Attr}^{\mathbb{G}}_{\text{Adam}}(G_i)$, which is equal to $W_{\text{Adam}}(\mathbb{G}_i)$ since $W_{\text{Eve}}(\mathbb{G}_i) = \emptyset$, Adam plays a winning strategy in $\mathbb{G}_i$.

This strategy is winning: a play consistent with it either eventually enters $W_{\text{Eve}}(\mathbb{G}_i)$ or it remains forever in $\text{Attr}^{\mathbb{G}}_{\text{Adam}}(G_i)$, which implies in both cases that it does not satisfy Rabin.

Question 5: Assume that $W_{\text{Eve}}(\mathbb{G}_i) \neq \emptyset$. Define $\mathbb{G}'$ the subgame of $\mathbb{G}$ where we remove $\text{Attr}^{\mathbb{G}}_{\text{Eve}}(W_{\text{Eve}}(\mathbb{G}_i))$. Show that $W_{\text{Adam}}(\mathbb{G}) = W_{\text{Adam}}(\mathbb{G}')$.

Solution 5: We show $W_{\text{Adam}}(\mathbb{G}) \subseteq W_{\text{Adam}}(\mathbb{G}')$. We actually show the contrapositive: $W_{\text{Eve}}(\mathbb{G}') \subseteq W_{\text{Eve}}(\mathbb{G})$. We construct a strategy σ in $\mathbb{G}'$:

- On $\text{Attr}^{\mathbb{G}}_{\text{Eve}}(W_{\text{Eve}}(\mathbb{G}_i)) \setminus W_{\text{Eve}}(\mathbb{G}_i)$, we play the attractor strategy to reach $W_{\text{Eve}}(\mathbb{G}_i)$.
- On $W_{\text{Eve}}(\mathbb{G}_i)$, we play a winning strategy in $\mathbb{G}_i$.
- On $W_{\text{Eve}}(\mathbb{G}')$, we play a winning strategy in $\mathbb{G}'$.

This strategy is winning.

We now show $W_{\text{Adam}}(\mathbb{G}) \supseteq W_{\text{Adam}}(\mathbb{G}')$. Let τ be a winning strategy in $W_{\text{Eve}}(\mathbb{G})$. It induces a winning strategy in $\mathbb{G}'$: indeed if Adam uses τ in $\mathbb{G}'$, then the play remains in $W_{\text{Adam}}(\mathbb{G}')$ so it is winning.

Question 6: Combining all of the above, construct an algorithm for solving Rabin games and evaluate its complexity as a function of n, m, d.

Question 7: Prove or disprove (for both players): Rabin is positionally determined.

Solution 7: Rabin is positionally determined for Eve: looking back at the proofs, we only construct positional winning strategies. It is not positionally determined for Adam, he needs memory. Example: Adam needs to see infinitely many times left and infinitely many times right, so he needs to alternate.

Exercise 2

The goal of this exercise is to prove the following theorem: every prefix-independent submixing objective is positionally determined for Eve over finite arenas. We say that Ω is submixing if:

$$
\begin{array}{llcllllllll}
\text{if} & \rho_1 & = & \rho_1^0 & & \rho_1^1 & & \cdots & \rho_1^\ell & & \cdots \notin \Omega \\
\text{and} & \rho_2 & = & & \rho_2^0 & & \rho_2^1 & \cdots & & \rho_2^\ell & \cdots \notin \Omega, \\
\text{then:} & \rho_1 \bowtie \rho_2 & = & \rho_1^0 & \rho_2^0 & \rho_1^1 & \rho_2^1 & \cdots & \rho_1^\ell & \rho_2^\ell & \cdots \notin \Omega.
\end{array}
$$

Question 1: Prove or disprove: Parity is submixing. Recall that:

$$\text{Parity} = \{\rho \in [1,d]^\omega : \max\inf(\rho) \text{ is even}\},$$

where $\inf(\rho)$ is the set of colours appearing infinitely many times in ρ.

Solution 1: It is submixing: the maximal priority appearing infinitely many times in $\rho_1 \bowtie \rho_2$ is the maximum among ρ_1 and among ρ_2, so if both are odd, then it is odd as well.

The proof is by induction on the following quantity:

$$\sum_{v \in V_{\text{Eve}}} \left|\left\{(v, v') : (v, v') \in E\right\}\right| - |V_{\text{Eve}}|$$

Question 2: Prove the base case of the induction.

Solution 2: Since we assume that every vertex has an outgoing edge, the base case is when each vertex of Eve has only one successor. In that case Eve has only one strategy and it is positional, so the property holds.

We now do the induction step. Let Ω be a prefix-independent submixing objective. Let $\mathbb{G}$ a finite game with objective Ω and assume that Eve has a winning strategy from v_0. Let v a vertex of Eve having outdegree at least two, let us partition the outgoing edges of v into $E_1 \cup E_2$. We consider the two games $\mathbb{G}_1$ and $\mathbb{G}_2$ where we restrict the outgoing edges of v to E_1 and to E_2, respectively.

Question 3: Prove that Eve has a winning strategy in $\mathbb{G}_1$ or in $\mathbb{G}_2$. Hint: reason by contradiction to obtain winning strategies for Adam in $\mathbb{G}_1$ and in $\mathbb{G}_2$, and construct a strategy for Adam in $\mathbb{G}$.

In the proof, highlight the argument relying on prefix-independence!

Solution 3: We claim that Eve has a winning strategy in either $\mathbb{G}_1$ or $\mathbb{G}_2$. Let us assume towards contradiction that this is not the case: then there exist τ_1 and τ_2 two strategies for Adam which are winning in $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively. We construct a strategy τ for Adam in $\mathbb{G}$ as follows: it has two modes, 1 and 2. The initial mode is 1, and the strategy simulates τ_1 from the mode 1 and τ_2 from the mode 2. Whenever v is visited, the mode is adjusted: if the outgoing edge is in E_1^v then the new mode is 1, otherwise it is 2. To be more specific: when simulating τ_1 we play ignoring the parts of the play using mode 2, so removing them yields a play consistent with τ_1. The same goes for τ_2.

Consider a play π consistent with σ and τ. Since σ is winning, the play π is winning. It can be decomposed following which mode the play is in:

$$\begin{array}{llllll} \text{mode } 1 & \overbrace{v_0 \cdots v}^{\pi_1^0} & & \overbrace{v \cdots v}^{\pi_1^1} & & \cdots \\ \text{mode } 2 & & \underbrace{v \cdots v}_{\pi_2^0} & & \underbrace{v \cdots v}_{\pi_2^1} & \cdots \end{array}$$

where $\pi_1 = \pi_1^0 \pi_1^1 \cdots$ is consistent with τ_1 and $\pi_2 = \pi_2^0 \pi_2^1 \cdots$ is consistent with τ_2. Since τ_1 and τ_2 are winning strategies for Adam, π_1 and π_2 do not satisfy Ω.

There are two cases: the decomposition is either finite or infinite. If it is finite we get a contradiction: since π is winning and Ω is prefix independent any suffix of π is winning as well, contradicting that it is consistent with either τ_1 or τ_2 hence cannot be winning.

In the second case we get a contradiction using the submixing property: neither π_1 nor π_2 satisfy Ω, yet their shuffle π does.