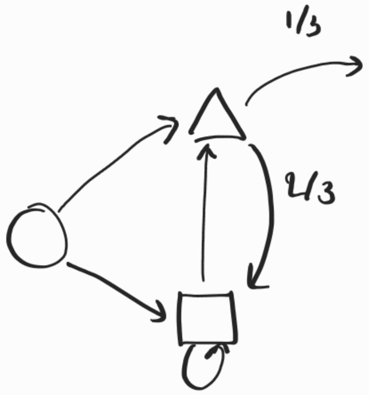
STOCHASTIC GAMES



○ Max σ

□ Min τ

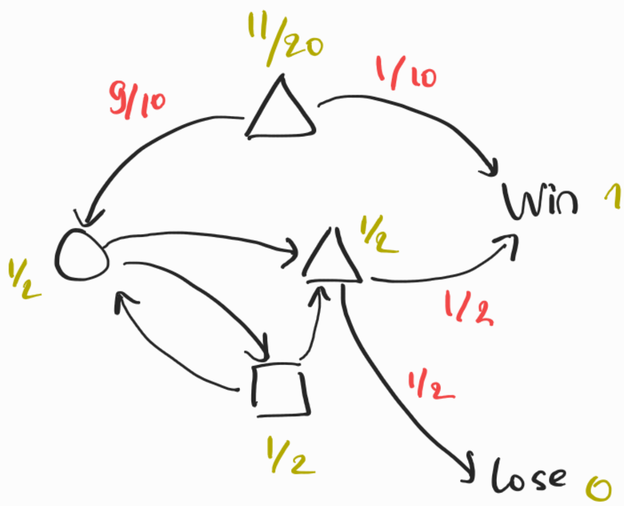
△ Random

$$V = V_{Max} \uplus V_{Min} \uplus V_{Random}$$

STOCHASTIC REACHABILITY GAMES

Reach: we fix $Win \subseteq V$

$$Reach = \{ \rho = v_0 v_1 \ldots \in V^\omega : \exists i \ \ v_i \in Win \}$$



$$\mathrm{val} : V \longrightarrow [0,1] \qquad \text{value function}$$

$$\mathrm{val}^{\sigma,\tau}(v) = \mathbb{P}_v^{\sigma,\tau}(\mathrm{Reach}(\mathrm{Win}))$$

$$= \mathbb{E}_v^{\sigma,\tau}[\mathbb{1}_{\mathrm{Reach}(\mathrm{Win})}]$$

$$\mathrm{val}_{Max}(v) = \sup_\sigma \inf_\tau \mathrm{val}^{\sigma,\tau}(v)$$

$$\leqslant \inf_\tau \sup_\sigma \mathrm{val}^{\sigma,\tau}(v) = \mathrm{val}_{Min}(v)$$

Theorem for stochastic reachability games:

- $\mathrm{val}_{Max} = \mathrm{val}_{Min} = \mathrm{val}_*$
- pure positional optimal strategies exist

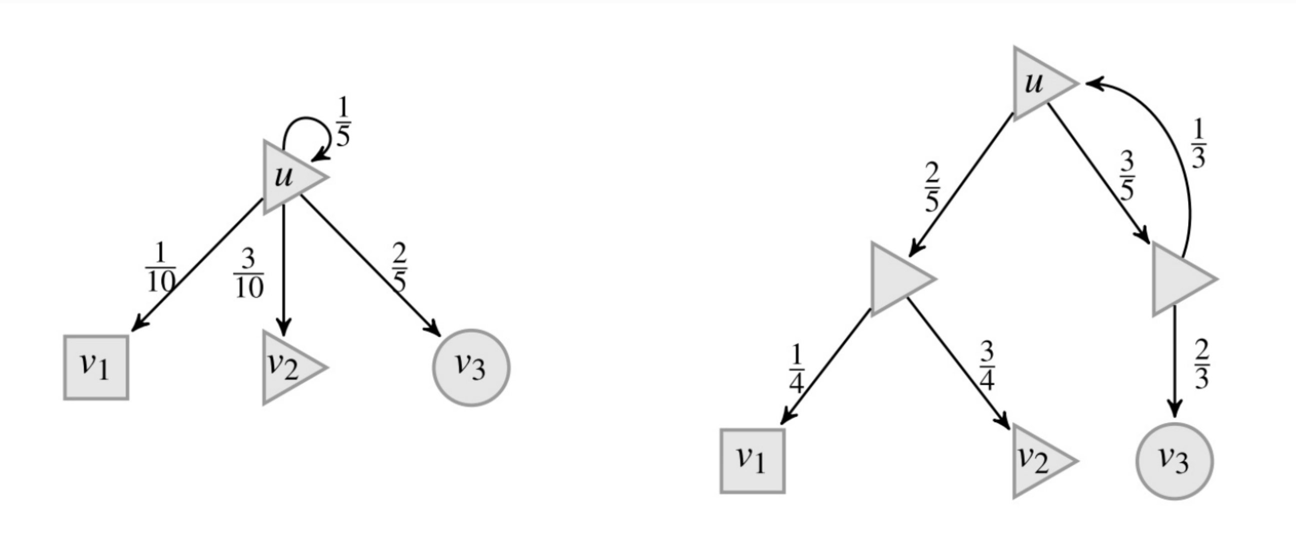
$$\begin{cases} \sigma : V_{Max} \longrightarrow E \\ \tau : V_{Min} \longrightarrow E \end{cases}$$

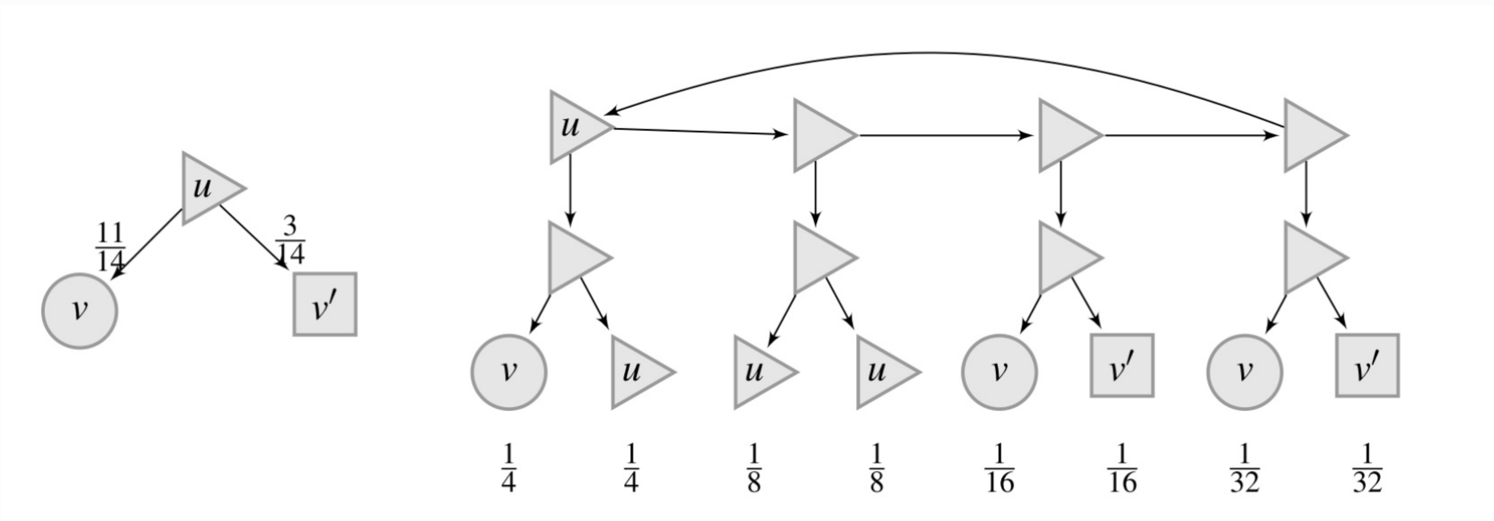
SIMPLE STOCHASTIC GAMES

stochastic reachability games where:

- outdegree = 2
- probabilities are binary : $\frac{1}{2}$ / $\frac{1}{2}$

REDUCTION





Theorem (Knaster - Tarski)

$(\mathcal{L}, \leq)$ complete lattice

$\Phi : (\mathcal{L}, \leq) \longrightarrow (\mathcal{L}, \leq)$

monotonic $\quad x \leq y \Rightarrow \Phi(x) \leq \Phi(y)$

preserves suprema: $\quad \sup_n \Phi(x_n) = \Phi(\sup_n x_n)$

(1) Φ has a unique least fix point: $\Phi(x) = x$

(2) the least fixpoint is also the least prefixpoint: $\Phi(x) \leq x$

(3) $x_0 = \perp \qquad x_{k+1} = \Phi(x_k)$

$\lim_k x_k$ is the least fixed point of Φ

$$Y = \{ \mu : V \longrightarrow [0,1] \}$$

$\mu \leq \mu'$ if $\forall v \in V \ \mu(v) \leq \mu'(v)$

$(Y, \leq)$ is a complete lattice

$$\Phi : Y \longrightarrow Y$$

$$\Phi(\mu)(u) = \begin{cases} 1 & \text{if } u \in Win \\ \max \{ \mu(v) : u \to v \} & u \in V_{Max} \\ \min \{ \mu(v) : u \to v \} & u \in V_{Min} \\ \sum_v \delta(v \mid u)\, \mu(v) & u \in V_{Random} \end{cases}$$

Φ monotonic, preserves suprema

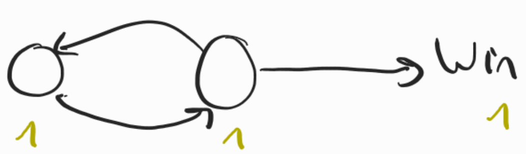
Theorem: $val^* =$ Least Fixed Point of Φ

if $\forall u \in V_{Min}$

$\tau(u) \in \operatorname{argmin} \{ val^*(v) : u \to v \}$

then τ is optimal

⚠ False for Max:



$\mu_0 = 0 \qquad \mu_{k+1} = \Phi(\mu_k)$

$\mu_* = \lim_k \mu_k \qquad \mu_* = LFP(\Phi)$

$Reach_{\leq k} = \{ \rho = v_0 v_1 \dots \in V^\omega : \exists i \leq k \ \ v_i \in Win \}$

Claims

(1) $\mu_k = \sup_\sigma \inf_\tau \mathbb{P}^{\sigma,\tau}(Reach_{\leq k})$

(2) $\mu_* \leq val_{Max}$

[val_{Max} is a prefixed point of Φ

(3) $val_{Min} \leq \mu_*$

[invariant : $\forall \sigma \ \forall k \ \forall u$

$\mathbb{E}_{u,\sigma}\left[\mu^*(\pi_k)\right] \leq \mu^*(u)$

POSITIONAL DETERMINACY FOR MAX

Fix $\lambda \in (0,1]$

$$\mathrm{QReach}_\lambda(\rho) = \begin{cases} 0 & \text{if } \rho \notin \mathrm{Reach} \\ \lambda^i & \text{if } i = \text{first } v_i \in \mathrm{Win} \end{cases}$$

$$Y = \{ \mu : V \longrightarrow [0,1] \}$$

$$\Phi_\lambda : Y \longrightarrow Y$$

$$\Phi_\lambda(\mu)(u) = \begin{cases} 1 & \text{if } u \in \mathrm{Win} \\ \lambda \max \{ \mu(v) : u \to v \} & u \in V_{\max} \\ \lambda \min \{ \mu(v) : u \to v \} & u \in V_{\min} \\ \lambda \sum_v \delta(v \mid u)\, \mu(v) \end{cases}$$

Φ_λ monotonic, preserves suprema, and contracting

Theorem (Banach)

$(X, \|\cdot\|)$ complete space

$\Phi : X \to X$

λ-contracting : $\quad \|\Phi(x) - \Phi(y)\| \leq \lambda \|x - y\|$

Φ has a unique fixed point x_*

$\forall x_0 \qquad \lim_k \Phi^k(x_0) = x_*$

$$\|\Phi^k(x_0) - x_*\| \leq \frac{\lambda^k}{1-\lambda} \|\Phi(x_0) - x_0\|$$

Step 1 : study $QReach_\lambda$

σ optimal $\iff$ $\forall u \in V_{Max}$ $\sigma(u) \in \arg\max \{ val_\lambda(v) : u \to v \}$

τ optimal $\iff$ $\forall u \in V_{Min}$ $\sigma(u) \in \arg\min \{ val_\lambda(v) : u \to v \}$

Step 2 : $\lim_{\lambda \to 1} QReach_\lambda = Reach$

$\exists (\lambda_k)_k \longrightarrow 1$ all use the same σ pure positional

Claim: σ optimal for Reach

(1) $\forall \mu : V \to [0,1]$ $\forall \lambda$

$$\| \Phi_\lambda(\mu) - \Phi(\mu) \| \leq 1 - \lambda$$

(2) val_{λ_k} fixed point of Φ_{λ_k}

So $\| val_{\lambda_k} - \Phi(val_{\lambda_k}) \| \leq 1 - \lambda_k$

(3) $\lim_k val_{\lambda_k}$ is a fixed point of Φ

So $\lim_k val_{\lambda_k} \geq val_*$

(4) $\mathbb{P}_u^{\sigma,\tau}(Reach) \geq \mathbb{P}_u^{\sigma,\tau}(QReach_{\lambda_k}) \geq val_{\lambda_k}(u)$

$k \to \infty$

$\forall \tau$ $\mathbb{P}_u^{\sigma,\tau}(Reach) \geq \lim_k val_{\lambda_k}(u) = val_*(u)$

So : σ is optimal for Reach